Michael Goldberg, G. A. Harter, J. R. Kline, P. A. Knedler, H. M. Lufkin, Edith A. McDougle, D. L. McDonough, A. E. Meder, Jr., H. H. Mitchell, F. W. Owens, Helen B. Owens, C. J. Rees, George Rosengarten, J. A. Roulton, I.M. Sheffer, J. A. Shohat, A. H. Wilson.

At the business meeting the following officers were chosen for next year: Chairman, Professor Tomlinson Fort, Lehigh University; Secretary, P. A. Caris, University of Pennsylvania; Program Committee, Professors Fort (ex-officio), Clawson and Frink. The next meeting of the Section will be held on Saturday, November 28, 1931, at the University of Pennsylvania.

The following papers were presented:

1. "On orthogonal Tchebycheff polynomials" by Dr. J. A. Shohat, University of Pennsylvania.

2. "A polar reciprocation of the complete quadrilateral" by Professor J. W. Clawson, Ursinus College.

3. "Some remarks on non-analytic functions" by Professor I. M. Sheffer, Pennsylvania State College.

4. "Almost-periodic functions" by Professor Tomlinson Fort, Lehigh University. Abstracts of the papers follow:

1. Definition, existence and uniqueness of orthogonal Tchebycheff polynomials; distribution of their roots. Examples: polynomials of Legendre, Jacobi, Laguerre, Hermite. Some minimum properties of Tchebycheff polynomials. Applications to: (a) expansion of functions in series of orthogonal polynomials; (b) mechanical quadratures; (c) polynomials in general.

2. Professor Clawson described a polar reciprocation of a complete quadrilateral and some of its related points and lines with respect to a circle having the focal (Steiner, Miquel) point of the quadrilateral for center. The quadrangle having for vertices the centers of the four circles circumscribing the triangles determined by three of the sides of the quadrilateral has for its reciprocal a second quadrilateral. The new quadrilateral is inversely similar to the original quadrilateral, and is homothetic to the inverse of the original quadrilateral with respect to the same circle; these three quadrilaterals possess a common focal point and common incentric lines. Moreover the quadrangle whose vertices are the circumcenters of the second quadrilateral is the polar reciprocal of the original quadrilateral.

3. Non-analytic functions of a complex variable are, as their name suggests, those functions for which a unique derivative does not exist (in general) at a point. Rather to each direction at a point a unique derivative exists, and (Kasner) when all directions at a point are taken the resulting derivative-locus is a circle. The property of conformality no longer holds for such functions but to each direction at a point corresponds a second direction forming a pair of dual directions, such that dual directions are transformed conformally. Especially important are the orthogonal dual pair and the self-dual directions. Dual pairs are geometrically intimately related to the Kasner circle. There is also an algebraic aspect of these functions in terms of certain matrices of the second order;

and a general process of differentiation of matrices when applied to these matrices yields known formulas for differentiation of non-analytic functions.

4. Professor Fort discussed the fundamental notions of the papers by Harald Bohr published in volumes 45, 46, and 47 of *Acta Mathematica*. He touched briefly upon work of other authors as Besicovitch, Weyl, and Wiener.

P. A. CARIS, *Secretary*

CORRELATION COEFFICIENTS AND TRANSFORMATION OF AXES

By MARIAN A. WILDER, University of Minnesota

It is well known that correlation coefficients can be interpreted as cosines of angles. One form of interpretation comes directly from the fundamental formulas of analytical geometry in space of n dimensions.[1] Another form of interpretation is based on the simplification of the exponent in the equation for the normal correlation surface.[2] This treatment in terms of reduction of the quadratic form is based on the assumption that the distribution is normal, while the first method does not require that the distribution be restricted in form at all. The purpose of this paper is to show that a calculation essentially equivalent to that involved in the reduction of the quadratic form can be carried through for the completely general distribution without assuming that it be normal.

Let $(x_1, x_2, \cdots, x_n)$,$(y_1, y_2, \cdots, y_n)$ and $(z_1, z_2, \cdots, z_n)$ be three sets of n real numbers each, in which the variables are deviations from their respective arithmetic means, or such that

$$\sum x_k = \sum y_k = \sum z_k = 0.$$

(Throughout this paper the sign $\sum$ stands for a summation from $k=1$ to $k=n$.) We shall also assume that no set consists entirely of zeros, and that the three sets are not linearly dependent.

For each k, let

$$\zeta_k = z_k,$$
$$\eta_k = y_k - \beta\zeta_k,$$
$$\xi_k = x_k - \alpha_1\eta_k - \alpha_2\zeta_k,$$

where

$$\beta = \sum y_k\zeta_k / \sum \zeta_k^2, \qquad \alpha_1 = \sum x_k\eta_k / \sum \eta_k^2, \qquad \alpha_2 = \sum x_k\zeta_k / \sum \zeta_k^2.$$

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[1] D. Jackson, *The elementary geometry of function space*, this Monthly, vol. 31 (1924), pp. 461–471; *The geometry of frequency functions*, this Monthly, vol. 31 (1925), pp. 63–73; *The Theory of Approximation*, New York (1930), pp. 154–169; *The trigonometry of correlation*, this Monthly, vol. 31 (1924), pp. 275–280.

[2] James McMahon, *Hyperspherical goniometry and its application to correlation theory for N variables*, Biometrika, vol. 15 (1923), pp. 173–208; Karl Pearson, *Excess and its relation to correlation*, Philosophical Transactions, Series A, vol. 197 (1901), pp. 18–20.

Then ξ, η, ζ satisfy the conditions $\sum \xi_k \eta_k = \sum \xi_k \zeta_k = \sum \eta_k \zeta_k = 0$, as is readily seen by substitution.

Now let

$$\lambda_k = \xi_k (n/\sum \xi_k^2)^{1/2},$$
$$\mu_k = \eta_k (n/\sum \eta_k^2)^{1/2},$$
$$\nu_k = \zeta_k (n/\sum \zeta_k^2)^{1/2}.$$

Then λ, μ, ν satisfy the conditions of orthogonality and have standard deviations equal to one. We can now solve for x, y, and z in terms of λ, μ, ν, giving equations of the form

$$\begin{aligned} x_k &= a_1\lambda_k + a_2\mu_k + a_3\nu_k, \\ y_k &= \quad b_2\mu_k + b_3\nu_k, \\ z_k &= \quad\quad c_3\nu_k. \end{aligned} \tag{1}$$

It will be noted that the coefficients a_1, b_2, c_3 are positive.

If the numbers $(\lambda_k, \mu_k, \nu_k)$ are plotted as sets of coordinates with respect to a system of rectangular axes, the quantities (x_k, y_k, z_k) may be regarded as coordinates of the same points in an oblique coordinate system, with suitable scales of measurement along the axes. The equations of transformation,

$$\begin{aligned} x &= a_1\lambda + a_2\mu + a_3\nu, \\ y &= \quad b_2\mu + b_3\nu, \\ z &= \quad\quad c_3\nu, \end{aligned} \tag{2}$$

are adjusted to the values of the original x's, y's, and z's so that the points having the oblique coordinates (x_k, y_k, z_k) are represented by rectangular coordinates with zero correlations and unit standard deviations. We show that the angles of the oblique coordinate system then are related to the coefficients of correlation of the original x's, y's, and z's.

Specifically, we calculate the angles between the normals to the planes $x=0, y=0, z=0$. The direction cosines of the normals are

$$\begin{array}{ccc} a_1/d_1, & a_2/d_1, & a_3/d_1; \\ 0\ , & b_2/d_2, & b_3/d_2; \\ 0\ , & 0\ , & c_3/d_3; \end{array}$$

where

$$d_1 = (a_1^2 + a_2^2 + a_3^2)^{1/2},\ d_2 = (b_2^2 + b_3^2)^{1/2},\ d_3 = c_3.$$

As to algebraic sign, the first three direction cosines are those of the ray from the origin toward the point with the rectangular coordinates (a_1, a_2, a_3). By the first of equations (2) the x-coordinate of this point is $a_1^2 + a_2^2 + a_3^2 > 0$. So the normal points toward the side of the plane $x=0$ where x is positive. Similarly, the other normals are on the same side of the corresponding coordinate planes

as the positive y and z axes respectively. The cosine of the angle between the planes is given by the sum of the products of the direction cosines. Hence the cosine of the angle between the planes $x=0$ and $y=0$ is $(a_2b_2+a_3b_3)/d_1d_2$; between the planes $x=0$ and $z=0$ is a_3c_3/d_1d_3; and between the planes $y=0$ and $z=0$ is b_3c_3/d_2d_3; the angle in each case being that between the normals specified. But, from equations (1), together with the conditions satisfied by the λ's, μ's, and ν's,

$$
\begin{aligned}
(1/n)\textstyle\sum x_k^2 &= a_1^2 + a_2^2 + a_3^2 = d_1^2, \\
(1/n)\textstyle\sum y_k^2 &= b_2^2 + b_3^2 = d_2^2, \\
(1/n)\textstyle\sum z_k^2 &= c_3^2 = d_3^2, \\
(1/n)\textstyle\sum x_k y_k &= a_2b_2 + a_3b_3, \\
(1/n)\textstyle\sum x_k z_k &= a_3c_3, \\
(1/n)\textstyle\sum y_k z_k &= b_3c_3.
\end{aligned}
$$

Therefore the cosines of the angles between the normals are

$$
\begin{aligned}
\textstyle\sum x_k y_k / (\sum x_k^2 \sum y_k^2)^{1/2} &= r_{xy}, \\
\textstyle\sum x_k z_k / (\sum x_k^2 \sum z_k^2)^{1/2} &= r_{xz} \\
\textstyle\sum y_k z_k / (\sum y_k^2 \sum z_k^2)^{1/2} &= r_{yz}.
\end{aligned}
$$

Geometric interpretations of coefficients of partial correlation, double correlation, etc., then follow as in the other treatments referred to.

These facts are perhaps rather closely suggested by the formulas of McMahon's article, but the logical development there is based throughout on the assumption that the distribution is normal.

It may be noticed that equations (1) are of the form

$$
\begin{aligned}
x_k &= a_1\lambda_k + a_2\mu_k + a_3\nu_k, \\
y_k &= b_1\lambda_k + b_2\mu_k + b_3\nu_k, \\
z_k &= c_1\lambda_k + c_2\mu_k + c_3\nu_k.
\end{aligned}
$$

The fact that $b_1=c_1=c_2=0$ in the particular transformation given here is immaterial. Representations of the same form, with different coefficients, are possible in terms of an infinite variety of sets of λ's, μ's, and ν's, where the new λ, μ, ν axes would be a rotation of the old ones.

By the use of a suitable notation, the method can be carried over without essential change to the case of m statistical variables and m dimensions.

A CALCULUS OF VARIATIONS PROBLEM WHOSE EXTREMALS ARE PARABOLAS

By PEARL BIERMAN and H. A. SIMMONS, Northwestern University

1. *Introduction.* The purpose of this paper is to discuss a problem which is highly illustrative of much of the elementary theory of the calculus of variations. In fact our problem is a good companion of the catenary problem in that they both illustrate essentially the same features of the calculus of variations theory. However, our problem involves *simpler algebra* than does the catenary problem, and defines *algebraically* many quantities which in the catenary problem are necessarily defined *transcendentally*. For example, in this paper conjugate points are found algebraically by solving two simultaneous quadratics and the MacNeish curve is found in a rational algebraic form.

We endeavor here to make a discussion that is essentially a parallel of the much appreciated treatment which Professor Bliss[1] has given of the catenary problem. We hope that the use of this method in connection with the problem here treated will be valuable not only to beginning students of the calculus of variations but also to other individuals who have found the algebra of this subject rather complicated and have not enjoyed a genuine appeal of the elementary portion of the calculus of variations theory.

The integral with which we are principally concerned is the case $n=1/2$ of the integral of Euler,[2]

$$I = \int_{x_1}^{x_2} y^n(1 + y'^2)^{1/2}dx,$$

and belongs therefore to a family of calculus of variations problems which includes such famous ones as the catenary ($n=1$) and the brachistochrone ($n=-1/2$). As indicated in footnote 2 above, the integral which we use most has received some attention, though we believe our treatment of it is the first one that describes satisfactorily the resemblance of this problem to the catenary problem.

2. *Statement of the problem.* Suppose that light travels in the xy-plane and that in this plane the index of refraction, $n(x, y)$, is directly proportional to the square root of the distance of the point (x, y) from the x-axis; then the velocity v of the light satisfies a relation of the form $v(x, y)=kn(x, y)=k/y^{1/2}$, where k is a suitably chosen constant. Further, let us call a curve $y=y(x)$ *admissible* if it has $y\geqq 0$, is continuous, and has a continuously turning tangent except possibly at a finite number of points. Our problem, then, is to find among all admissible curves through $1(x_1, y_1)$, $2(x_2, y_2)$, one along which light will pass from 1 to 2 in the shortest time. Since $ds/dt=k/y^{1/2}$, we have when $x_2>x_1$:

[1] Cf. Carus Mathematical Monograph No. 1, *Calculus of Variations*, by G. A. Bliss. We are indebted to Professor Bliss for a few suggestions concerning the present problem.

[2] Cf. Bolza's *Vorlesungen Über Variationsrechnung*, p. 145, ex. 11, and Goursat's *Cours d'Analyse Mathématique*, (1915), vol. 3, p. 555.

$$t = k\int_{x_1}^{x_2} y^{1/2}(1 + y'^2)^{1/2}dx.$$

Obviously $t>0$ and so $k>0$; we seek then to minimize the integral

$$I = \int_{x_1}^{x_2} y^{1/2}(1 + y'^2)^{1/2}dx \tag{1}$$

by properly determining the function $y=y(x)$, which satisfies the relations

$$y_1 = y(x_1),\ y_2 = y(x_2). \tag{2}$$

Later we shall consider the case $x_2 \geqq x_1$ and admit a general class of parametric curves.

3. *Proof that the minimizing arc is a parabola.* The integrand function in (1) is $f \equiv y^{1/2}(1+y'^2)^{1/2}$, and so $f_{y'y'} = [y^{1/2}(1+y'^2)^{-3/2}] > 0$ for all points above the x-axis. Hence if we consider only admissible curves which lie in the region $y>0$, every solution of Euler's equation for our problem will have continuous curvature and will therefore not have a corner; such an arc will frequently be referred to as an *extremal.* From the expression for I, y could not be negative; the case in which $y=0$ at one or more points of the interval $x_1 \leqq x \leqq x_2$ will be discussed later. Since the curves under consideration have continuous curvature, Euler's differential equation, which must be satisfied (Cf. Bliss, p. 49),[3] has the consequence $f-y'f_{y'}=c$, or $y'=(y-c^2)^{1/2}/c$, where $c\neq 0$ is a constant of integration. Hence the extremals for our problem when $x_2>x_1$ and $y>0$ are defined by[4]

$$\pm\, 2(\alpha y - \alpha^2)^{1/2} = x - d,\ \alpha \equiv c^2 > 0; \tag{3}$$

or, by the equivalent of (3),

$$4\alpha(y - \alpha) = (x - d)^2. \tag{4}$$

The extremals are therefore parabolas with axes parallel to the y-axis and with the x-axis as directrix. Of (3) and (4), we shall in the future use the equation which best serves our convenience. We express the above result in

Theorem 1. If 1 and 2 are two given points in the half-plane $y>0$ and if $y=y(x)$ is an admissible arc E_{12} joining them and is a minimum time curve for our problem, then E_{12} is a single arc (without corners) of one of the parabolas (4).

4. *The number of parabolas of family* (4) *which pass through two given points.* If a parabola of (4) passes through 1 and 2, then by (2),

$$4\alpha(y_1 - \alpha) = (x_1 - d)^2,\ 4\alpha(y_2 - \alpha) = (x_2 - d)^2, \tag{5}$$

from which

---

[3] We shall use this method of referring to the first Carus Monograph.

[4] c is real in $f-y'f_{y'}=c$ since $y', f, f_{y'}$ are supposed to be real functions.

$$d = -2\alpha(y_2 - y_1)(x_2 - x_1)^{-1}\tfrac{1}{2}(x_1 + x_2). \tag{6}$$

From (5) and (6), we obtain two pairs of values for d, α. Setting $\gamma = 4y_1y_2 - (x_2 - x_1)^2$, we may express the solutions for α as follows:

$$\alpha_1 = \tfrac{1}{4}(x_2 - x_1)^2(y_1 + y_2 + \sqrt{\gamma})^{-1}, \quad \alpha_2 = \tfrac{1}{4}(x_2 - x_1)^2(y_1 + y_2 - \sqrt{\gamma})^{-1}. \tag{7}$$

The corresponding values for d_1, d_2 are of course to be obtained by substitution from (7) in (6). The number of parabolas of (4) that pass through 1 and 2 then depends upon the character of γ. We express our result in

Theorem 2. The exact number of parabolas of family (4) that pass through two fixed points 1 and 2 is zero when $\gamma < 0$, one when $\gamma = 0$, and two when $\gamma > 0$.

5. *The one-parameter family of parabolas through a given point.* In the two-parameter family of extremals (3), there is a one-parameter family which passes through 1 (Cf. Bliss, p. 92) and is given by

$$y = y_1 + \tfrac{1}{4}\alpha^{-1}[(x - x_1)^2 \pm 4(x - x_1)(\alpha y_1 - \alpha^2)^{1/2}] \quad (a \neq 0). \tag{8}$$

From the manner in which (8) was obtained, it must contain all of the parabolas of (3) that pass through 1, and therefore all of them that pass through 1 and 2. Hence (8) must be satisfied by the values x_2, y_2, α_i $(i = 1, 2)$. This requires that the $-$ sign before the radical in (8) be used when $\alpha = \alpha_1$; when $\alpha = \alpha_2$, it requires the $+$ sign if $\sqrt{\gamma} > 2y_1$ and the $-$ if $\sqrt{\gamma} < 2y_1$; when $\sqrt{\gamma} = 2y_1$, the sign is immaterial. We shall refer to these facts as the sign restrictions on (8). To distinguish carefully between the properties of the parabolas of (8) that are determined by α_1 and those that are defined by α_2 is one of our important tasks.

To obtain the envelope of the curves (8), we eliminate α between (8) and the equation $y_\alpha = 0$:

$$y_\alpha = \tfrac{1}{4}\alpha^{-2}\{-(x - x_1)[(x - x_1)(\alpha y_1 - \alpha^2)^{1/2} \pm 2\alpha y_1]\} = 0 \quad (\alpha \neq 0). \tag{9}$$

Cancellation of the factor $x - x_1$ from the numerator of (9) merely loses the point $1(x_1, y_1)$ of the envelope; for all other points of the envelope, we have

$$\alpha = y_1(x - x_1)^2/[4y_1^2 + (x - x_1)^2]. \tag{10}$$

Hence α is a single-valued function of x along the envelope, and there by theorem 2, $\gamma = 0$. Using this fact in requiring that (8) be satisfied by the value of α that is given in (10), we find the equation of the sought envelope (apart from the point 1) to be

$$y = \tfrac{1}{4}y_1^{-1}(x - x_1)^2, \quad (x \neq x_1). \tag{11}$$

Where it would be trivial to mention 1 as a part of the envelope, we shall refer to (11) as the envelope G of family (8). The result which is now evident we express in

Theorem 3. The envelope of the one-parameter family of extremals (8) through the point 1 is the parabola (11). It is not one of the parabolas (8).

The last two theorems together justify

Corollary 1. The exact number of parabolas of family (4) that pass through two given points 1 and 2 is zero when 2 is below the envelope G, one when 2 is on G, and two when 2 is above G.

6. *The Lindelöf construction for the conjugate point on a parabola of* (8). Since equation (4) is equivalent to

$$y = \alpha\{1 + \tfrac{1}{4}[(x - d)\alpha^{-1}]^2\} = \alpha\phi[(x - d)\alpha^{-1}]$$

we know from a theorem of Bolza[5] that there is a Lindelöf construction for the point conjugate to 1 on an extremal of family (8). This construction and the algebraic results that are needed in a proof of it are easily seen to be as follows. Denoting by E a parabola of (8) with a negative slope at 1 and with a parameter value α, we draw the tangent T_1 to E at 1 (Fig. 1). It intersects the x-axis at $[x_1+\alpha y_1(\alpha y_1-\alpha^2)^{-1/2}, 0)]$ At this point we construct a perpendicular T_2 to T_1. The line T_2 is tangent to G and E at

$$[x_1 + 2\alpha y_1(\alpha y_1 + \alpha^2)^{-1/2},\ \alpha y_1(y_1 - \alpha)^{-1}].$$

Thus when $\alpha=\alpha_1, \alpha_2$ [Cf. (7)], the $-$ sign being used before the radical in (8), we obtain two parabolas E', E'', and the points conjugate to 1 on them are

$$P_1\left[x_1 + \frac{2y_1(x_2 - x_1)}{2y_1 + \sqrt{\gamma}},\ \frac{y_1(x_2 - x_1)^2}{(2y_1 + \sqrt{\gamma})^2}\right],$$

$$P_2\left[x_1 + \frac{2y_1(x_2 - x_1)}{2y_1 - \sqrt{\gamma}},\ \frac{y_1(x_2 - x_1)^2}{(2y_1 - \sqrt{\gamma})^2}\right],$$

respectively. On the interval $x_1<x<\infty$, a parabola of (8) whose slope is positive at 1 lies entirely above the parabola $y=y_1+\frac{1}{4}y_1^{-1}(x-x_1)^2$ and therefore has no contact with G.

We summarize the above results in

Theorem 4. A parabola E of (8) has on it a point conjugate to 1 and to the right of 1 if and only if the slope of E at 1 is negative. Then the conjugate point is P_1 or P_2, as defined above, according as $\alpha=\alpha_1$ or $\alpha=\alpha_2$. In both cases, the conjugate point is located by drawing two tangents T_1, T_2 to E, T_1 being tangent to E at 1 and T_2 perpendicular to T_1 where T_1 meets the x-axis.

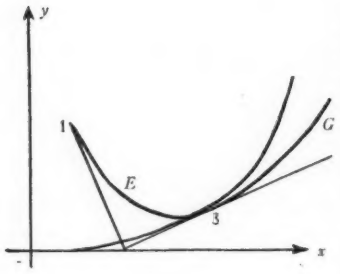

FIG. 1

7. *Further properties of family* (8). In accordance with a well known theorem on linear differential equations, we know that since $f_{y'y'}>0$ for $y>0$, the complete solution

[5] Cf. O. Bolza, Bulletin of American Mathematical Society, vol. 18 (1911), p. 107.

(4) of Euler's differential equation for our problem contains exactly one extremal through the point 1 in every direction. Hence family (8) has this property. We shall observe this fact below in showing how a parabola of (8) with parameter value α varies with α.

Let $x=x_2$ be a fixed value greater than x_1; substitute α_1 for α in (8); and let α_1 increase from 0 to its maximum value

$$\{y_1(x_2 - x_1)^2[4y_1^2 + (x_2 - x_1)^2]\} = \tfrac{1}{4}(x_2 - x_1)^2(y_1 + y_2)^{-1}$$

on G. Then y_2 decreases continuously from ∞ (when $\alpha=0$) to $\frac{1}{4}(x_2-x_1)^2y_1^{-1}$ on G. While $y=y_2$ is so decreasing, the slope at 1 of the parabola with parameter value α_1 increases continuously from $-\infty$ to $-2y_1(x_2-x_1)^{-1}$. These limiting values of y' may be found as follows. At $x=x_1$, $dy/dx=-(\alpha y_1-\alpha^2)^{1/2}\,\alpha^{-1}$, whose limiting value as α approaches zero is $-\infty$; when $\alpha=\alpha_1=\frac{1}{4}(x_2-x_1)^2(y_1+y_2)^{-1}$, on G, the expression for y' becomes

$$-\left[\frac{y_1(x_2 - x_1)^2}{4(y_1 + y_2)} - \frac{(x_2 - x_1)^4}{16(y_1 + y_2)^2}\right]^{1/2} \div \frac{(x_2 - x_1)^2}{4(y_2 + y_2)}$$

$$= -\frac{[4y_1^2 + 4y_1y_2 - (x_2 - x_1)^2]^{1/2}}{x_2 - x_1} = -\frac{(4y_1^2 + \gamma)^{1/2}}{x_2 - x_1} = \frac{-2y_1}{x_2 - x_1}.$$

Similarly one can show that if the $-$ sign is used before the radical in (8) while $\alpha=\alpha_2$, then as α_2 increases continuously from its value on the envelope G, namely $\frac{1}{4}(x_2-x_1)^2(y_1+y_2)^{-1}$, to y_1, the value of y_2 will increase continuously from $\frac{1}{4}(x_2-x_1)^2y_1^{-1}$ to $y_1+\frac{1}{4}(x_2-x_1)^2y_1^{-1}$, respectively. While $y=y_2$ is so increasing, the slope at 1 of the parabola with parameter value α_2 increases continuously from $-2y_1(x_2-x_1)^{-1}$ to 0. Then if we use the $+$ sign before the radical of (8) and let $\alpha=\alpha_2$ decrease continuously from y_1 to 0, $y=y_2$ increases from $y_1+\frac{1}{4}(x_2-x_1)^2y_1^{-1}$ to ∞, respectively. While y_2 is so increasing, the slope at 1 of the parabola with parameter value α_2 increases from 0 when $\alpha_2=y_1$ to ∞ when $\alpha_2=0$.

Now, as in §6, let E', E'' be particular parabolas of (8) defined by the parameters α_1, α_2, respectively. When $\alpha=\alpha_1$ and 2 is above G, the equations (8) and (11) have a simultaneous solution (x, y) such that $x_1<x<x_2$, as can be seen by examination of the abscissa of the point P_1 mentioned in §6. Hence E' touches G between 1 and 2. When $\alpha=\alpha_2$ and 2 is above G, equations (8) and (11) have no such solution, as can be seen from the abscissa of the point P_2 of §6. Hence E'' does not touch G between 1 and 2. If 2 is on G, (8) and (11) of course have a simultaneous solution at 2 regardless of which parameter $\alpha_i(i=1, 2)$ is used.

We summarize the results of this section in

Theorem 5. In every direction through 1 there passes one, and only one, parabola of (8). When 2 is above G and $x_2>x_1$, every parabola E' defined by α_1 touches G between 1 and 2, while no parabola E'' defined by α_2 so touches G.

If $x_2 > x_1$ and 2 is on G, E', and E'' coincide and touch G at 2. If $x_2 > x_1$ and 2 is below G, we have the case of Theorem 2 when $\gamma < 0$.

8. *The envelope theorem and Jacobi's condition.* To make the investigation which is next in order, we need to use the notion of a *field* and the Hilbert invariant integral. A *field* is a region of the xy-plane through every point of which there passes one, and only one, extremal of a one-parameter family of extremals all of which meet a fixed curve. If we consider only the portion of each extremal in (8) from the point 1 to the point to the right of 1 where it touches G, the region of the xy-plane for which $x > x_1$ and above G will form a field F since a portion of one of the two extremals of (8) [Cf. Cor. 1] through any point 2 above G and to the right of 1 will have been disregarded and since all of the extremals of (8) meet G. This implies that when the parameter value α in (8) is assigned and the proper sign before the radical of (8) is used, the slope function $p(x,y) = dy/dx$, obtained by differentiating (8), is uniquely defined at every point (x, y) of F. The Hilbert invariant integral for our problem is then defined by the equation

$$I^* = \int_a^b [f(y, p)dx + (dy - pdx)f_{y'}], \tag{12}$$

where a and $b > a$ are abscissas of points of F, $f(y, p) = y^{1/2}(1+p^2)^{1/2}$, y is defined by (8) when in this equation a particular α is used in accordance with our sign restrictions (Cf. §5), and dx, dy are differentials of the functions $x = x(t)$, $y = y(t)$, $t_1 \leqq t \leqq t_2$, that define in F a curve C which passes through two points with abscissas a, b and has a continuously turning tangent except possibly at a finite number of points. By adjoining the single point $1(x_1, y_1)$ to F, we form an improper field F'. After suitable assumptions[6] are made as to the values of $p(x, y)$, $\alpha(x, y)$ at 1, it is known that I^* has in F' the two fundamental properties that it has in F, namely: it is independent of the path of integration; its value along an extremal of F' from $x = a$ to $x = b > a$ is equal to $I(E_{ab})$.

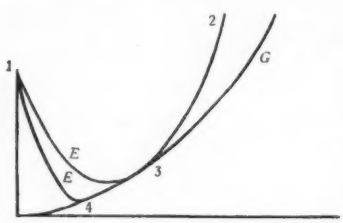

FIG. 2

Since G is a parabola and has a branch projecting to the left from any one of its points 3, between 1 and 2, our problem is one for which customary procedure (Cf. Bliss, pp. 140–141) with the integrals I and I^* in F' gives the following envelope theorem.

Theorem 6. If two parabolas E_{14} and E_{13} of family (8) touch G at 4 and 3, respectively (Fig. 2), where $x_4 < x_3$, then the time curves $E_{14} + G_{43}$ and E_{13} give equal values to I: $I(E_{14}) + I(G_{43}) = I(E_{13})$.

From Theorem 6 and the fact $f_{y'y'} > 0$ when $y > 0$, the following corollary is obvious.

[6] Cf. *Lectures on the Calculus of Variations*, by O. Bolza, p. 83, footnote 2.

Corollary 2. (Jacobi's condition). If a parabolic arc E_{12} is to afford a minimum time curve for our problem, the point of contact 3 of E_{12} with G must not lie on the arc E_{12}.

From Corollary 2 we know that the extremal E' mentioned in Theorem 5 can never afford a minimum time curve for our problem. In fact, since G_{43} is not an extremal arc, it is known that G_{43} of the last theorem can be replaced by an arc C_{43} such that $I(E_{14})+I(C_{43})<I(E_{13})$ [Cf. Bliss, p. 141]. Hence if our problem has a solution, it must be afforded by the curve E'' mentioned in Theorem 5. We shall investigate the minimizing properties of E'' in fields of the types which are described in Bliss, §§40 and 44.

9. *Construction of a field in a V-shaped region.* We have seen that the arc E_{12} of the parabola E'' mentioned in theorem 5 contains no point conjugate to 1. Let $d=d_2$, when $\alpha=\alpha_2$ [Cf. (6), (7)]. Then the equation of E'', taken from (4), is

$$y-\alpha_2=\tfrac{1}{4}\alpha_2^{-1}(x-d_2)^2. \tag{13}$$

Let 0 be a point of E'' [Fig. 3] to the left of 1, but so close to 1 that the point 3 of E'' which is conjugate to 0 is still to the right of 2. The tangents to E'' at 0 and 3 meet on the x-axis at a point 4. The transformation

$$x-x_4=\alpha_2\alpha^{-1}(X-x_4),\quad y=\alpha_2\alpha^{-1}Y, \tag{14}$$

stretches the plane along radial lines through 4 in such a way that every point (x, y) is replaced by a point (X, Y) whose distance from 4 is $\alpha\alpha_2^{-1}$ times that of (x, y). Substituting the values x, y of (14) into (13), we find that the points (x, y) of E'' are transformed into points (X, Y) which satisfy the equation

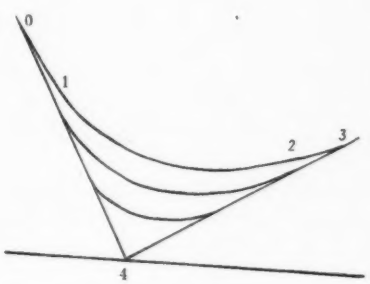

FIG. 3

$$y-\alpha=\tfrac{1}{4}\alpha^{-1}[x-x_4+\alpha\alpha_2^{-1}(x_4-d_2)]^2. \tag{15}$$

For a fixed value of α (15) represents a parabola of family (4) with parameters α and $d=x_4-\alpha\alpha_2^{-1}(x_4-d_2)$. Letting α vary, we see that (15) represents a one-parameter family of parabolas containing E_{03} for $\alpha=\alpha_2$ and tangent to the lines joining 4 to 0 and 3. Through each point (x, y) of the V there passes a unique extremal of family (15), and all members of the family intersect any straight line which radiates from 4 through the V. We have now constructed a field F which is analogous to the field described in Bliss, §40.

10. *Properties of the field functions.* Before we can make a sufficiency proof we must assure ourselves that certain continuity properties are possessed by

the field functions which enter into the Hilbert invariant integral (12), namely $\alpha=\alpha(x, y)$, which in (15) is the ordinate of the vertex of the parabola whose parameter value is α, and

$$p = p(x, y) = \partial y(x, \alpha)/\partial x = y'[x, \alpha(x, y)] = \tfrac{1}{2}\alpha^{-1}[x - x_4 + \alpha\alpha_2^{-1}(x_4 - d_2)],$$

which is the slope of the extremal E of (15) through the point (x, y) of F. It is obvious geometrically that α, p are uniquely defined in F. We wish to prove them *continuous* in F. If we prove the property for α, we shall have it also for p, as is seen from the definition of p. Differentiating (15) with respect to α and making a few simplifications in the result, we find

$$y_\alpha = \tfrac{1}{4}\alpha_2^{-2}[4\alpha_2^2 + (x_4 - d_2)^2] - \tfrac{1}{4}\alpha^{-2}(x - x_4)^2;\ \alpha \neq 0.$$

We assume $\alpha\neq 0$ here and thus exclude the vertex 4 of the V from consideration; this does not place an additional restriction upon our problem as we have supposed from the beginning of this paper that $y>0$ when our extremals are of the form $y=y(x)$. We see from the fact that y_α vanishes for only one admissible value of α, namely

$$[\alpha_2^2(x - x_4)^2\{4\alpha_2^2 + (x_4 - d_2)^2\}^{-1}]^{1/2},$$

and changes signs at this value, that y_α vanishes only on the boundary of V. From this fact, it can be proved by a method used in Bliss [Cf. pps. 107–108], that in the interior of V the function $\alpha(x, y)$ is continuous and has continuous partial derivatives of the first order with respect to x and y. Furthermore, by methods which are used in Bliss [pp. 60–62], it can be proved that α is continuous on the boundary of F except at the vertex. We omit these proofs because they are of the same character as the analogous proofs in Bliss.

11. *The sufficiency theorem.* That the extremal arc E_{12} of E'' is the minimizing admissible time curve $y=y(x)$ in F and joining the points 1, 2 can be proved in the same way that an analogous proof is made in Bliss [pp. 108–109]; we merely state the result:

Theorem 7. An admissible arc $y=y(x)$ on the interval $x_1\leqq x\leqq x_2$ with $x_2>x_1$, in the half-plane $y>0$, joining two fixed points 1 and 2, and affording a minimum time curve for our problem must be a single arc, without corners, of one of the parabolas (4) and must not have a point of contact with the envelope G, $y=\frac{1}{4}y_1^{-1}(x-x_1)^2$ of the one-parameter family of parabolas (8) through 1. Giving α the value α_2 of (7) and determining the corresponding value of $d=d_2$, by (6), we obtain an arc E_{12} with these two properties. If then F is one of the V-shaped regions described above, containing E_{12} in its interior and bounded by two tangents to E_{12} which meet on the x-axis, and if light travels as we have supposed, it will pass from 1 to 2 in a shorter time along E_{12} than it will along any other arc C_{12} with equations $x=x(t)$, $y=y(t)$, $t_1\leqq t\leqq t_2$, which lies in F, passes through 1, 2 and has a continuously turning tangent on t_1t_2 except possibly at a finite number of points.

This result can also be obtained by specializing a general theorem concerning *regular problems* for which the envelope analogous to G is of a particular type [Cf. Bliss, p. 161]. Indeed the conditions denoted by I, III, and IV′ in Chapter V of Bliss are both necessary and sufficient for a strong relative minimum in our problem.

Hereafter we shall refer to comparison arcs C_{12} of the type that was mentioned in Theorem 7 as *parametric arcs*.

12. *Discontinuous solutions.* The argument given in Bliss, §43, applies to our problem. We state our result in

Theorem 8. If a vertical straight line E_{12} has its upper end-point 1 in common with a parametric arc C_{13} whose length equals that of E_{12}, as shown in Fig. 4, then the time required by light of the type that we hypothesize to pass from 1 to 2 along E_{12} is less than that required to pass from 1 to 3 along C_{13} unless C_{13} coincides with E_{12}.

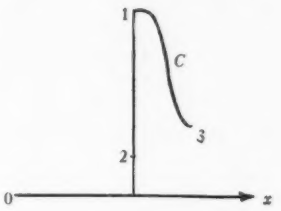

FIG. 4

Immediate consequences of Theorem 8 are the corollaries below.

Corollary 3. When 1 and 2 are in the same vertical line $x=x_1$, the segment of this line that joins 1 and 2 affords an absolute minimum time curve for our problem.

Corollary 4. If 1 and 2 are not in the same vertical line and if C_{12} is a parametric arc of length $\geqq y_1+y_2$, joining 1 and 2 and lying in the region $y\geqq 0$, the time required by light of the type that we consider to pass from 1 to 2 along the line segments 13, 34, 42, whose equations are $x=x_1$, $y=0$, $x=x_2$, respectively, is at most as great as the corresponding time over C_{12}. The arc e_{12} consisting of the segments 13, 34, 42 has a neighborhood N of the type shown in Fig. 5 in which e_{12} is an absolute minimum time curve for our problem.

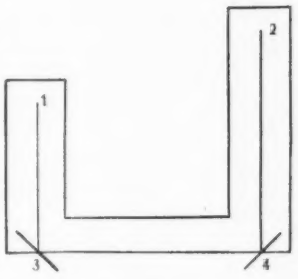

FIG. 5

In corollary 4 the time over 34 was taken to be zero since $v=ky^{-1/2}$.

The broken line e_{12} of Corollary 4 is the Goldschmidt discontinuous solution for our problem. We shall give further information about it later. It is obvious that in order to pass light from 1 to 2 along e_{12}, one would need to place at 3 and 4 [Fig. 5] mirrors inclined at angles of 135° and 45°, respectively, with the positive x-axis. These mirrors would need to be of a special character; that is, they should either have the same index of refraction as the medium in which

the light has been supposed to travel, or they should be so constructed that light would not penetrate into them to a depth $\neq 0$.

13. *The minimum in a second type of field.* The following theorem can be proved by the method which is used in Bliss, §§44–45.

Theorem 9. If E_{12} is the arc from 1 to 2 of the parabola E'' found above, joining 1 with 2 and having on it no point conjugate to 1 except possibly at 2, the time required by the light which we hypothesize to pass from 1 to 2 along E_{12} is less than the corresponding time along every other parametric arc, joining 1 and 2, and, except possibly at 2, lying entirely above the envelope G of the one parameter family of parabolas through 1, the line $x=x_1$ counting as one of the parabolas.[7]

14. *The absolute minimum.* Several further results concerning the Goldschmidt discontinuous solution of our problem are entirely analogous to results which are given in Bliss, §46, and are obtainable by the methods of that section. We merely state the results in question:

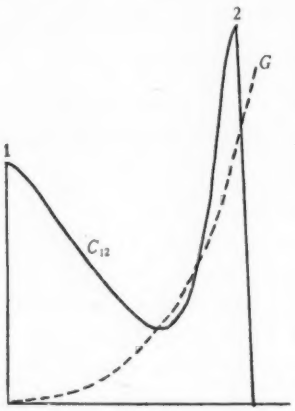

FIG. 6

Theorem 10. The time required by the light that we hypothesize to pass over every parametric arc C_{12} distinct from e_{12} and having a point 5 in common with the envelope G, as shown in Fig. 6, is greater than the time required on e_{12}.

Corollary 5. When there are fewer than two parabolas joining the points 1 and 2 the Goldschmidt solution e_{12} always furnishes an absolute minimum.

Corollary 6. When 2 is above G the parabolic arc E_{12} of Theorem 9 and e_{12} both furnish minima in sufficiently small neighborhoods, and the one over which our light passes in the shorter time provides an absolute minimum.

To distinguish the cases in which E_{12} gives the absolute minimum from those in which e_{12} gives this minimum, we shall obtain in a rational algebraic form the equation of a curve on which $I(e_{12})=I(F_{12})$ [Cf. the curve H in Fig. 7 below]. This curve will be called the MacNeish curve because it is an analog of a curve that was found by Mr. MacNeish[8] for the catenary problem.

When 2 is at 1, $I(E_{12})=0$ while

[7] The continuity of the extremal integral

$$I=\int_{x_1}^{x_3}[y(x,\alpha)]^{1/2}\,[1+y'^2(x,\alpha)]^{1/2}\,dx$$

when a point $3(x_3, y_3)$ varies on the interior and boundary of the field F of theorem 9 can be proved here by the method that is used in Bliss, §45.

[8] MacNeish, Annals of Mathematics, vol. 7 (1905), p. 72.

$$I(e_{12}) = 2\int_{y_1}^{y_2} [y(x, \alpha)]^{1/2} dy > 0, \ (ds = dy),$$

so that the difference $I(E_{12}) - I(e_{12}) < 0$. For every position of 2 above or on G, except on the line $x = x_1$ where 2 was taken in §10, we have

$$I(E_{12}) - I(e_{12}) = \int_0^{s_2} y^{1/2} ds - \tfrac{2}{3}(y_1^{3/2} + y_2^{3/2}).$$

As 2 moves from 1 along a fixed parabola E of family (8), the derivative $y_2^{1/2}(1 - dy_2/ds_2)$ of the difference $I(E_{12}) - I(e_{12})$ is positive, since $|dy_2/ds_2| < 1$. When 2 is on G, $I(E_{12}) > I(e_{12})$ by theorem 10. Hence there is exactly one position 2 on the parabola E at which $I(E_{12}) = I(e_{12})$. We ask, then, what is the locus of points (x_2, y_2) which satisfy the equation

$$\int_{x_1}^{x_2} [y(x, \alpha)]^{1/2} [1 + y'^2(x, \alpha)]^{1/2} dx = \tfrac{2}{3}(y_1^{3/2} + y_2^{3/2}), \tag{16}$$

where y is defined by (8) when $\alpha = \alpha_2$, which is the parameter value that defines the minimizing arc E_{12} of Theorem 7. Consequently the integrand of (16) is equal to

$$\alpha_2^{-3/2}[\alpha_2 y_1 + \tfrac{1}{4}(x - x_1)^2 \pm (x - x_1)(\alpha_2 y_1 - \alpha_2^2)^{1/2}],$$

the ambiguity in sign before the radical being only apparent, as we shall presently see. Using this expression for the integrand in (16) and carrying out the integration there, we find

$$I(E_{12}) = \alpha_2^{-3/2}[\alpha_2 y_1 (x_2 - x_1) + \tfrac{1}{12}(x_2 - x_1)^3 \pm \tfrac{1}{2}(x_2 - x_1)^2(\alpha_2 y_1 - \alpha_2^2)^{1/2}]. \tag{17}$$

Now substituting in (17) for α_2 its value $\frac{1}{4}(x_2 - x_1)^2 (y_1 + y_2 - \sqrt{\gamma})^{-1}$ [Cf. (7)], we find, after simplification,

$$I(E_{12}) = 2y_1[y_1 + y_2 - \sqrt{\gamma}]^{1/2} + \tfrac{2}{3}(y_1 + y_2 - \sqrt{\gamma})^{3/2} \pm [y_1 + y_2 - \sqrt{\gamma}]^{1/2}[4y_1^2 - 4y_1\sqrt{\gamma} + \gamma]^{1/2}.$$

Since

$$[4y_1^2 - 4y_1\sqrt{\gamma} + \gamma]^{1/2} = \sqrt{\gamma} - 2y_1 \text{ when } \sqrt{\gamma} > 2y_1,$$

and

$$= 2y_1 - \sqrt{\gamma} \text{ when } 2y_1 > \sqrt{\gamma},$$

and since the $+$ or $-$ sign in front of the radical of (8) is used according as $\sqrt{\gamma} > 2y_1$ or $\sqrt{\gamma} < 2y_1$, respectively, [Cf. §5], we have in all cases

$$I(E_{12}) = 2y_1[y_1 + y_2 - \sqrt{\gamma}]^{1/2} + \tfrac{2}{3}(y_1 + y_2 - \sqrt{\gamma})^{3/2} - (2y_1 - \sqrt{\gamma})[y_1 + y_2 - \sqrt{\gamma}]^{1/2}.$$

Using this result as the left member of (16), dropping the subscript 2, cancelling the factor $\frac{2}{3}$ from both members of the resulting equation, and letting $\gamma_1 = 4yy_1 - (x - x_1)^2$, we obtain

$$[y + y_1 - \sqrt{\gamma_1}]^{1/2}[y + y_1 + \tfrac{1}{2}\sqrt{\gamma_1}] = y^{3/2} + y_1^{3/2}.$$

Squaring both members of this equation, we obtain, after some simplification,

$$3(x - x_1)^2(y + y_1) - 8y^{3/2}y_1^{3/2} = \gamma_1^{3/2}.$$

Squaring this equation and simplifying again, we obtain

$$(18)\quad 9(x - x_1)^2(y + y_1)^2 - 48(y + y_1)(yy_1)^{3/2} + 48y^2y_1^2 - 12yy_1(x - x_1)^2 + (x - x_1)^4 = 0.$$

Transposing the second term of (18), collecting the other terms of the left member as a quadratic in $(x-x_1)^2$, and squaring the resulting equation, we obtain the rational form of the equation of the *MacNeish curve* for our problem:

$$(19)\quad \{(x - x_1)^4 + [9(y^2 + y_1^2) + 6yy_1](x - x_1)^2 + 48y^2y_1^2\}^2 = 2304(y_1 + y)^2y^3y_1^3.$$

By equating to zero the group of terms of lowest degree in the two variables $x-x_1$, y, we find that $y^3=0$ is the equation of three coincident tangents to the curve (19) at the point $(x_1, 0)$. To show that the MacNeish curve has no point except $(x_1, 0)$ in common with the envelope G, we proceed as follows. Set $y=\frac{1}{4}y_1^{-1}(x-x_1)^2$ in (19), cancel from both members of the resulting equation the factor $(x-x_1)^4$, and write $u\equiv(x-x_1)^2$. The resulting equation is

$$(9/16y_1^2)^2u^4 + (63/16y_1^2)u^3 + (179/8)u^2 + 63y_1^2u + 81y_1^2 = 0,$$

which has no positive root u.

Two more facts about the curve (19) are desired, namely that for every $x>x_1$ there is exactly one real value of y, and that y increases as $x>x_1$ increases. Solving (18) for $(x-x_1)^2$, we obtain

$$(20)\quad (x - x_1)^2 = \tfrac{1}{2}[-9(y^2 + y_1^2) - 6yy_1 + \{[9(y^2 + y_1^2) + 6yy_1]^2 + 192(yy_1)^{3/2}(y + y_1 - \sqrt{y}\sqrt{y_1})\}^{1/2}],$$

in which the + sign has been taken before the radical because $(x-x_1)^2>0$ for values of x that are being considered. After extracting the square root of both members of (20) and using the + sign before the newly introduced radical, we find that x is a single valued function of y:

$$(21)\quad x - x_1 = \left[\frac{-9(y^2+y_1^2)-6yy_1+\{9[(y^2+y_1^2)+6yy_1]^2+192(yy_1)^{3/2}(y+y_1-\sqrt{y}\sqrt{y_1})\}^{1/2}}{2}\right]^{1/2}$$

That $x>x_1$ increases with y can be shown best by differentiating equation (20) with respect to y and simplifying the result. As the proof is not necessarily simple, we give it in a few steps. Since $x-x_1>0$, $dx/dy>0$ if

$$(22)\quad [9(y^2 + y_1^2) + 6yy_1](3y + y_1) + 8y_1\sqrt{y}\sqrt{y_1}(5y + 3y_1 - 4\sqrt{y}\sqrt{y_1}) > (3y + y_1)\{[9(y^2 + y_1^2) + 6yy_1]^2 + 192(yy_1)^{3/2}(y + y_1 - \sqrt{y}\sqrt{y_1})\}^{1/2}.$$

Since $5y+3y_1>4\sqrt{y}\sqrt{y_1}$, both members of (22) are positive. Hence, after squaring both members of (22) and making some simplifications, we obtain the following necessary and sufficient condition that $dx/dy>0$ for $x>x_1$:

$$4\sqrt{y}\sqrt{y_1}(25y^2y_1+46yy_1^2+9y_1^3-40yy_1\sqrt{y}\sqrt{y_1}-24y_1^2\sqrt{y}\sqrt{y_1})$$
$$>(3y+y_1)(-9y^3-9y^2y_1-51yy_1^2-27y_1^3+12yy_1\sqrt{y}\sqrt{y_1}+36y_1^2\sqrt{y}\sqrt{y_1}),$$

and this can be reduced to

$$\sqrt{y}y_1(64y^2y_1+64yy_1^2)+2y^2y_1^2+36yy_1^3+36y^3y_1+27y^4+27y_1^4>0$$

which is true since $y_1>0$, and $y>0$ for $x>x_1$.

Now $dy/dx=1/(dx/dy)>0$ and so (19) defines y as a single-valued (real) function of x, which increases indefinitely as x so increases since the portion of the curve (19) for which $x>x_1$ lies entirely above G (Cf. Theorem 10).

If in equation (21) we give particular values to x_1, y_1, we can obtain for x numerical values (to any desired degree of accuracy) which correspond to assigned values of y. In obtaining the table of values below, which we used in plotting the accompanying MacNeish curve and the envelope G, we took $x_1=0$, $y_1=1$. The equation of G is then $y=\frac{1}{4}x^2$.

x	.79	1.36	1.88	2.24	2.51	2.73	2.92	3.08	3.22	3.36
H: y	.5	1	2	3	4	5	6	7	8	9
G: y	.16	.46	.88	1.25	1.58	1.86	2.13	2.37	2.59	2.82

As a check on the accuracy of the equation of the MacNeish curve, we shall consider two examples.

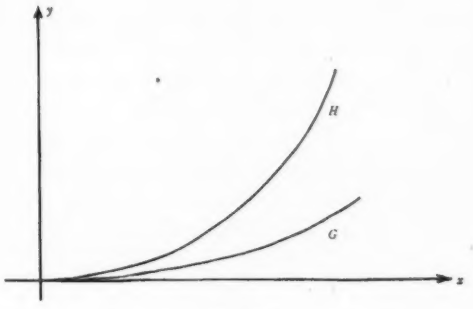

FIG. 7

Example 1. Compare $I(E_{12})$ and $I(e_{12})$ when 1 is at (0, 1) and 2 is at (1, 1).

We use the minimizing parabola of (8) through (0, 1) and (1, 1). From (7), $\alpha_2=\frac{1}{4}(2+\sqrt{3})$, and so the parabola of (8) is $y=1+(x^2-x)\,(2+\sqrt{3})^{-1}$. Consequently

$$I(E_{12}) = [2(2+\sqrt{3})^{3/2}]\int_0^1 (x^2 - x + 2 + \sqrt{3})dx$$

$$= \frac{2(2+\sqrt{3}-1/6)}{(2+\sqrt{3})^{-3/2}} < \frac{2}{3}(1+1) \equiv I(e_{12})$$

in accordance with our theory, since 2(1, 1) is above H [Fig. 7].

Example 2. Compare $I(E_{12})$ and $I(e_{12})$ when 1 is at (0, 1) and 2 is at (2, 2).

In this case $\alpha_2 = 1$ and the parabola which we desire from (8) has the equation $y = 1 + \frac{1}{4}x^2$. Hence

$$I(E_{12}) = \int_0^2 (1 + \tfrac{1}{4}x^2)dx = x + (x^3/12)\Big]_0^2 = 8/3 > \tfrac{2}{3}(y_1^{3/2} + y_2^{3/2}) = \tfrac{2}{3}(1+2\sqrt{2}),$$

or $I(E_{12}) > I(e_{12})$ in accordance with our theory since 2(2, 2) is between G and H.

Collecting the results of Theorem 10, the subsequent corollaries, and the above study of the MacNeish curve, we have

Theorem 11. When 2 is a point to the right of 1 and above the curve defined by (19), the Goldschmidt discontinuous solution e_{12}, joining 1 with 2, is a minimum time arc relative to other arcs of the parametric type (described in Theorem 7), joining 1 and 2, and lying in a sufficiently small neighborhood of e_{12}; but the absolute minimum time curve in this case is the unique parabolic arc E_{12}, joining 1 and 2, and having on it no contact point with the envelope G. When 2 is on the curve (19) and is at a point for which $x > x_1$ the times required by light of the type that we hypothesize to pass over the arc e_{12} and E_{12} are equal and are smaller than the corresponding time over every other arc of the parametric type referred to above. When 2 is between the curve (19) and the envelope G, and to the right of $x = x_1$, E_{12} provides a relative minimum time arc and e_{12} the absolute minimum. When 2 is on or below G, e_{12} is the only minimum time curve in the class of parametric arcs that we have considered, joining 1 and 2, and e_{12} furnishes the absolute minimum.

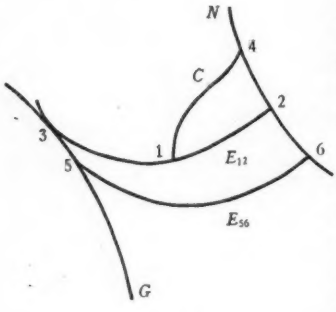

FIG. 8

15. *The case of one variable end-point.* The problem of finding the path along which light of the type that we hypothesize will pass from a given point 1 to a given curve N in the shortest time can be solved by argument of the type that is used in Bliss, §48. We merely state the result.

Theorem 12. If an admissible arc E_{12} (Cf. Fig. 8) minimizes the time of motion from a fixed point 1 to a fixed curve N, E_{12} is an arc of a parabola of the family $4(\alpha y - \alpha^2) = (x-d)^2$, intersecting N at right angles at 2, and having no point of contact with the envelope G of a one-parameter family of parabolas containing the parabola of

which E_{12} is an arc and intersecting N at right angles. Furthermore, an arc E_{12} which joins 1 to N and has these properties has a neighborhood F such that the time of motion along E_{12} is shorter than that along every other arc of the parametric type (mentioned in Theorem 7) which joins 1 to N and lies wholly in F.

The problem of finding the path along which light of the type that we hypothesize will pass from a given curve M to a given point 2 in the shortest time can be solved by the sort of argument that is used in Bliss, §32. We state the result in

Theorem 13. If an admissible arc E_{12} minimizes the time of motion from a fixed curve M to a fixed point 2 (Cf. Fig. 9) E_{12} is an arc of a parabola of the family $4(\alpha y-\alpha^2)=(x-d)^2$ such that the curve M at its intersection point 1 with E_{12} has a direction perpendicular to the tangent to E_{12} at 2, and such that E_{12} contains no point of contact with the envelope G of a one-parameter family of parabolas containing the parabola on which E_{12} is an arc and intersecting at right angles the curve N obtained by rotating M through an angle of 180° (in the xy-plane) on the center of the straight line segment joining 1 and 2. Furthermore, an arc E_{12} which joins M with 2 and has the other properties just mentioned has a neighborhood F such that the time of motion along E_{12} is shorter than that along every other arc of the parametric type (mentioned in Theorem 7) which joins M with 2 and lies entirely in F.

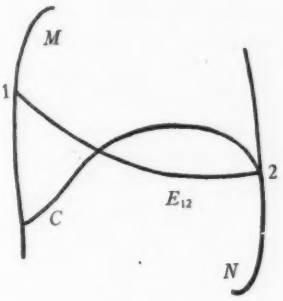

FIG. 9

16. *Geometrical construction for the focal point.* Since the extremals $4\alpha(y-\alpha)=(x-d)^2$ are of the form $y=\alpha\phi[(x-d)/\alpha]$, a well known geometrical construction for the focal point of the curve N on E_{12} (Cf. Fig. 8) can be applied. We omit the construction since it is the same as the corresponding construction for the catenary problem (Cf. Bliss, p. 125).

17. *Two variable end-points.* The problem of finding an arc joining two fixed curves and minimizing our time integral can be solved by argument of the type that is used in Bliss, §16. The result, we express in

Theorem 14. If an admissible arc E_{12} minimizes the time of motion from a given curve M to a second given curve N (Cf. Fig. 10), E_{12} is an arc of a parabola of the family $4(\alpha y-\alpha^2)=(x-d)^2$, M and N must intersect E_{12} at right angles at 1 and 2, respectively, and 1, 2 with the focal points 3, 4 of M, N, respectively, on E_{12} lie in the circular order 4312, no coincidences being allowed except that 4 may possibly fall upon 3. Further-

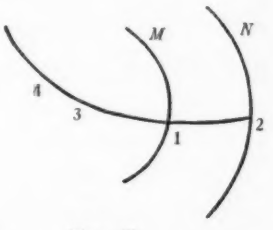

FIG. 10

more, an arc E_{12} which joins M to N and possesses the properties just mentioned has a neighborhood F such that the time of motion along E_{12} is shorter than that along every other arc of the parametric type (of Theorem 7) which joins M with N and lies entirely in F.

18. *Remarks on the corresponding problem in three dimensions.* Suppose that light travels in ordinary space of three dimensions; that x, y, z denote the rectangular coordinates of any point in this space; and that throughout the space the index of refraction $n(x, y, z)$ is directly proportional to the square root of the distance of the point (x, y, z) from the xz-plane. The velocity, v, of the light then satisfies a relation of the form $v(x, y, z) = kn(x, y, z) = k/\sqrt{y}$, where k is a suitably chosen constant. Let us call a curve $y = y(x)$, $z = z(x)$ *admissible* if it has $y \geqq 0$, is continuous, and has a continuously turning tangent except possibly at a finite number of points. Our problem then is to find among all admissible curves through two fixed points $1(x_1, y_1, z_1)$, $2(x_2, y_2, z_2)$, one along which light will pass from 1 to 2 in the shortest time.

Proceeding as in §2, we find that our problem is to minimize the integral

$$I = \int_{x_1}^{x_2} y^{1/2}[1 + y'^2 + z'^2]dx$$

by properly determining the functions $y(x)$, $z(x)$, which satisfy the relations $y_1 = y(x_1)$, $z_1 = z(x_1)$; $y_2 = y(x_2)$, $z_2 = z(x_2)$.

The integrand of the integral I is a special case of an integrand function $f(x, y, y', z, z')$ that has been used in calculus of variations theory.[9] Furthermore, in our problem $f_{y'y'}f_{z'z'} - f_{y'z'}^2 = y(1+y'^2+z'^2)^{-2} > 0$ when $y > 0$, so that in the region $y > 0$ our problem is *regular*, and the extremals admitted have no corners. Hence Euler's differential equations, which must be satisfied, have the consequences[10]

$$f - y'f_{y'} - z'f_{z'} = c_1, \quad f_{z'} = c_2;$$

or

$$1 = c_1[1 + y'^2 + z'^2]^{1/2}y^{-1/2}, \quad z' = c_2[1 + y'^2 + z'^2]^{1/2}y^{-1/2}.$$

Hence $z' = c_2c_1^{-1}$, $z = c_2c_1^{-1}x + c_3$, and we have the result: *the motion takes place in a plane perpendicular to the xz-plane.*

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[9] For example, Cf. Goursat, loc. cit., pp. 559–562.

[10] Cf. Goursat, loc. cit., p. 561.

ITALIAN CONTRIBUTIONS TO MODERN MATHEMATICS[1]

By ENRICO BOMPIANI, Rome, Italy

I accepted with pleasure the invitation of speaking to you on the Italian contributions to modern mathematics although I am perfectly aware of the difficulty of my task. It is such a wide subject that I cannot pretend to dominate it. Nor is it possible, by the nature of this report, to follow a rigorous classification of the matter from an historical viewpoint or by its content. It will therefore be sufficient to note, with some simple allusions to problems and the results, the boundaries in which such activity took place.

Before all it is necessary to define the limits of time to which I refer. I shall not speak to you of Archimedes of Syracuse nor of Zeno of Elea, whose contributions are also to be attributed to modern mathematics inasmuch as with the first we have reference to problems and methods of infinitesimal calculus, and with the second originated the logical discussion on continuity and discontinuity which precedes the modern theory of point-sets.

But it is perhaps necessary to mention here Leonardo Pisano, who, with his Liber Abbaci, brought into Italy the methods of calculation of the Arabians. The diffusion of this book in Europe, diffusion which was also a consequence of the political conditions then prevailing in Italy, gave the first impulse to the creation of our modern arithmetic and algebra.

At the beginning of the 16th century, the mathematical school of Bologna irradiated a new light on the scientific world with the discovery of the resolution of the equations of the third and fourth degree. With this discovery are connected the names of Scipione del Ferro, Ludovico Ferrari, Bombelli, Tartaglia of Brescia, Cardano of Milan. The question of the possibility of solving an algebraic equation of any order was answered by Palo Ruffini of Modena, who in the year 1799 demonstrated, before Abel, the impossibility of the algebraic solution of an equation of order greater than the fourth, giving at the same time the first germs of the theory of finite groups of substitutions, developed after him in its essential lines by Galois.

That the interest in pure arithmetic and algebra is still alive in Italy, may be proved by the researches of L. Bianchi on the modular group, on the theory of Dirichlet's and Hermitian forms, and on quadratic quaternary forms with integral rational coefficients; by the researches of Cipolla on asymptotic arithmetic; by the researches of Scorza and Cecioni on algebra.[2]

The 17th century marks the discovery of analytic geometry and of infinitesimal calculus.

The "methodus indivisibilium" of Cavalieri, which is yet referred to in modern treatises on the matter, and the numerous integrations of parabolas and

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[1] A lecture delivered at the Fourteenth Summer Meeting of the Mathematical Association of America at Providence, R. I., on Sept. 9, 1930.

[2] See the excellent treatise of G. Scorza: *Corpi numerici e algebre* (Principato, Messina, 1922).

hyperbolas of any order by Torricelli, testify not only their genius but also that the field was ripe for the symbolical systematization of Newton and Leibniz.

To the same period belongs the discovery of the infinite algorithm, like the continued fractions (for which Cataldi gave the formulas of computation of the successive convergents and the theorem on the best approximation), the infinite series, the logarithm. Mengoli, of Bologna, had a very exact idea of the definition of limit and of the different possibilities in the summation of a series,[3] Brunacci gave (in 1804) his famous transformation which is known today as the Theorem of Brunacci-Abel and which is so useful for the evaluation of series. In modern times Cesaro, one of our most original mathematicians, extended the idea of sum to the non-convergent series; this was the first step in the effective study of non-convergent series, developed particularly by Borel. G. Sannia (at Naples) has put in evidence that the Borel method of addition is only a particular case of an infinite chain of methods requiring steadily weaker conditions for the summability of a series. More recently Picone, at Naples, has given a general treatment of all summation methods, freed from initial restrictions which limited their possible applications; his pupil, Mammana, at Cagliari, is studying at present the algebraic operations on series summed with different methods of which, notwithstanding their importance for the applications, we know very little.

I shall also mention that Cesaro's method has given some splendid results, worked out by Féjer, in the approximate representation of real functions by trigonometric polynomials.

After the remarkable discoveries of the 17th and 18th centuries a revision of the principles on which mathematical knowledge lay was necessary as a condition for its further progress. This critical current concerned itself with the analysis of the principles of geometry, at the beginning of the 19th century, and had its outlet in the construction of non-euclidean geometries (Bolyai, Lobatchewski, Gauss and Riemann). The forerunner of these geometries was the Italian Jesuit, Saccheri, who, in a book entitled "Euclidis ab omni naevo vindicatus," endeavored to demonstrate the postulate of Euclid, but in reality he gave the first exposition, coherent and elementary, of the various types of non-euclidean geometries. Beltrami, above all, contributed to their acceptance, giving a tangible interpretation of them on the surfaces of constant curvature. When the interest in these constructions (whose critical purpose was completely achieved) was cooling, the same constructions acquired a new interest for the geometry of restricted relativity in the space-time of Einstein-Minkowski.

Part of the work of Veronese (although of later date) was devoted to the revision of the principles of geometry; he gave one of the first logical arrangements of postulates and the first example of non-archimedean geometry, found again later by Hilbert. Together with the name of Veronese there are to be mentioned,

[3] Many papers by Ettore Bortolotti, relating to the development of mathematics in Italy during this period, are collected in a volume: "Studi e ricerche sulla Storia della Matematica in Italia nei secoli XVI e XVII" (Zanichelli, Bologna, 1928).

for Italy, those of Burali-Forti and of Pieri, who assumed primitive notions different from those of Veronese.

An analogous revision of the foundations of the infinitesimal calculus, started by Bolzano and Cauchy, was brought about in the second half of the 19th century in a rigorous treatment of that theory. In this accomplishment have taken part in Italy, Dini, whose work on the functions of a real variable and on the Fourier's series is known everywhere and which points out for the first time fundamental conceptions of modern analysis; Ascoli and Arzelà,[4] whose work on systems of curves has furnished the rigorous basis for the existence theorems of differential and functional equations; Peano (and his school: Vailati, Vacca, Burali-Forti, Pieri, Padoa) to whom is due a classical formulation of the principles of arithmetic and the creation of a symbolic logic which is one of the most appreciated.

To the further development of this critical current in analysis may be also connected the name of Vitali, some of whose fundamental results are now considered classical, and the most recent contributions to the problem of integration of discontinuous functions by Tonelli and by Caccioppoli, who uses the new concept of limit given by Picone.[5]

To the classical works directed to the determination of the conditions of existence and uniqueness of integrals of ordinary differential equations belong those of Volterra, of Peano (whose method was recently found again by Perron and extended to partial differential equations) of Arzelà, Nicoletti and others. The question of the uniqueness of a solution in a given interval was answered by Lipschitz assuming one condition evidently too restrictive; recently I have given a general criterion of comparison between the integrals of two differential equations from which is deduced, in particular, the theorem of uniqueness of Osgood and Tamarkine; Tonelli and Perron have demonstrated again this theorem with different methods. Recent researches in this field are due to the school of Picone, whose young pupils Cimmino, Colucci, the younger Scorza and Caccioppoli have largely extended our knowledge in this field. The last one using, without knowing it, an idea already brought out by Birkhoff and Kellogg, has established, by very simple considerations of topology of functional spaces, a theorem of existence and of uniqueness for very wide classes of functional equations.

Perhaps the most conspicuous work in the beginning of the 19th century is the analytic mechanics of Lagrangia, from Torino, who has moulded on his principles a great part of the following production in applied mathematics. It is impossible to summarize even briefly the results attained in this field, even limiting ourselves to those given in Italy.[6] I shall mention therefore only the re-

[4] Arzelà introduced, except for the name, the concept of a normal family of functions, rediscovered later by Montel.

[5] See his "Lezioni di Analisi Infinitesimale" (published by the Circolo Matematico di Catania, Catania, 1923)—Some classical theories have received a new treatment in this book.

[6] A comprehensive treatise in analytical mechanics by Levi-Civita and Amaldi has been published by the Casa Zanichelli, Bologna.

searches of Levi-Civita and of his school (Bisconcini, Signorini, Armellini) on the problem of three bodies, which have prepared the ground for the result of Sundmann. To the classic mechanics belongs also the problem of motion of a body of variable mass, either by increase or by diminution. Levi-Civita has demonstrated, on the basis of two natural hypotheses, namely independence of effects and statistical isotropy, that the application of the principles of classical mechanics leads not to the equation of Lagrange (mass times acceleration equal to force) but to the theorem of quantity of motion (rate of change of the quantity of motion, or momentum, is equal to the force) which, as is known, remains still valid in the restricted relativity theory. The conclusions of Levi-Civita were applied by one of his pupils, Vranceanu, to the problem of two bodies of variable masses. This problem was already treated by Armellini with the use of higher analytical methods: Levi-Civita has tackled again the problem using the hypothesis of adiabaticity (not necessary in the solution of Armellini) and has made important applications to astronomy and has succeeded in assigning the conditions of minimum energy also in the case of revolving bodies. This is a consequence of a theory developed by Levi-Civita of the adiabatic invariants of differential systems of Liouville, in which are framed the theorems of Gibbs, Hertz and Burgers. Geppert, under the guidance of Levi-Civita, has extended the theory of the adiabatic invariants to more general differential systems.

A recent contribution in the field of classical analytical mechanics was given by Gugino who has introduced for a material system (whose constraints are without friction and independent of the time) a new scalar, the *cinetodynamic effect;* and has demonstrated that the natural motion of a system is characterized, compared with all possible motions satisfying the same conditions, by the fact that the cinetodynamic effect is a maximum.

To the problems of applied mathematics contributed actively in Italy Betti,[7] who with his famous reciprocity theorem gave impulse to the wide and deep studies on elasticity; Beltrami, Dini, Bianchi, Arzelà, Lauricella, Cerruti, Somigliana, Marcolango, Tedone, Burgatti, Almansi, Boggio; and particularly Volterra, who extended, in papers now classical, the method of Riemann to elastic vibrating bodies. Also to Volterra is due the creation of the theory of distortions of elastic multiply-connected bodies (a theory whose consequences were clearly verified by the experiments of Trabacchi and Corbino). Further in this group of works are to be mentioned the researches of Volterra on internal cyclical motions, of Almansi on the equilibrium of sands and of Signorini on reenforced concrete.

Also to applied mathematics belong the famous paper of Levi-Civita on waves, which has given a new impulse to the study of plane hydrodynamics (by a suitable use of analytic functions) and in which have amply participated Cisotti, Signorini, Colonnetti, Finzi, Pistolesi, Masotti and many others; the rigorous solution, given by Levi-Civita, of the problem of Airy on the progressive

[7] Betti is also to be remembered as one of the founders of analysis situs: to him is due the introduction of the "Betti numbers."

waves of permanent type in straight-channels and the extension of the theory to circular channels (as is necessary for the experimental verification) given by Geppert; the study of Masotti on the motions of a perfect liquid which take place in non-plane strata; and finally the extension of the theorem of Bernoulli to homogeneous viscous liquids which is due to Lelli.

Before closing this brief review of Italian contributions to analytical mechanics and applied mathematics, it is necessary that I recall at least the names of Giorgi and of Maggi for the critical analysis of its principles; of Armellini and Burgatti for the law of distribution of planets, of Marcolongo, Boggio and Burali-Forti, who have given a symbolism of vector-analysis which is one of the most logical and convenient for the various applications.[8]

Equations similar to those of classical mechanics are found in the study of the phenomena of economics; the founder of the statistics of these phenomena was Pareto from Genoa; his work was continued in Italy, also for dynamics, by Amoroso and Scorza.

The problems of applied mathematics as well as of geometry led from the very beginning to the study of partial differential equations and of analytic functions; in fact, many of the aforementioned researches may be considered as contributions to these fields. The greatest attention to the questions of existence and of uniqueness of a solution of a partial differential equation for given boundary conditions was aroused by the famous problem of Dirichlet. Riemann believed that he had solved it by a procedure which successive criticism has proved not sufficient. With different procedures the problem was solved by Neumann, Schwarz and Poincaré. But it was Arzelà who tried first (1897) to give rigorous foundation to the idea of Riemann; an attempt which only succeeded in our century through the work of Hilbert, B. Levi, Fubini, Lebesgue, Zaremba. The method of minimizing sequences of Fubini was recently used by Courant and the school of Göttingen.

On the existence question of differential equations of the second order, and of their connection with the integral and integro-differential equations, worked Nicoletti, Severini, E. E. Levi, Picone (whose fundamental identity, which can be used in many other problems, has received from Bôcher the name of identity of Picone) Mammana, Cimmino, and Tricomi who has examined the behavior of the integrals and the conditions of uniqueness of the solution of an equation of the 2nd order with non-constant coefficients in the vicinity of its parabolic curve.

Notwithstanding the interest of these results, the problem which interests most the physicist and the technician is the numerical evaluation of the solution, the existence of which has been previously demonstrated. In this connection some results of Picone and of his school are worthy to be mentioned: the rational formulas for the perturbed motion of projectiles (taking into account the variations of the physical and dynamical condition of the atmosphere; the

[8] Reference may be made to the treatise "Analisi Vettoriale" (published by Zanichelli, Bologna).

curvature and the rotation of the earth; etc.); the formulas establishing dominating values for the solutions of any linear equation of the second order of elliptic or parabolic type (with remarkable consequences regarding the boundary conditions which may be given to determine a solution), and also for some equations of mathematical physics of order greater than the second; the evaluation of the error in the approximate calculation of the solutions of the aforementioned equations. It should be remarked that this evaluation is made to depend only upon the knowledge of approximate values of the characteristic numbers and of dominant numbers of the solution and of some of its derivatives.

From this situation arises the importance of the problem of the evaluation of the characteristic numbers to which Picone is applying successfully a newly devised method in his institute at Naples.

In the theory of harmonic functions of two variables we shall also call attention to the existence theorem for the Neumann problem in arbitrarily connected domains, given by Picone; and also to some contributions, given by Picone and G. Ascoli, to the study of singularities of these functions.

The aforementioned researches refer to the functions of real variables; we will discuss them again when dealing with the functional calculus. But first, in order to conform to the historical development, it is necessary to refer to the theory of analytic functions.

Closely related to the work of Weierstrass and Mittag-Leffler were the remarkable researches of Pincherle; but we should also remember Betti, to whom are due (before Weierstrass gave his general theorem) the developments of several entire functions in infinite products; Morera, who has inverted the famous Cauchy Theorem; and Casorati who gave the first characteristic theorem on essential singularities, a theorem which precedes the famous ones of Picard-Landau. Vivanti's theorem on the determination of a singular point on the circumference of convergence of a series with non-negative coefficients is the first of its kind, and was followed by the brilliant researches of Phragmen, Lindelöf, Mandelbrojt and recently of the young Italian, Minetti.

To the theory of some notable classes of analytic functions, like elliptic, abelian, modular, automorphic functions, have contributed Casorati, Brioschi, Pascal, D'Ovidio, Bianchi, Pincherle, Fubini, Enriques, Severi, Bagnera, De Franchis, Scorza, Comessatti, Rosati and Spampinato. The theory of algebraic functions, considered as analytic functions, is flourishing in Italy from the geometric view-point, of which I shall have occasion to speak to you later.

To the analytic functions of several complex variables, the study of which we can say has just started, although several essential results of Weierstrass, of Poincaré, of Osgood and of Carathéodory are known, important papers have been dedicated by E. E. Levi with the study of the varieties which may be boundaries of such functions, by Levi-Civita with the study of the characteristic surfaces of the equations of monogeneity, and more recently by Severi.

To another branch of analysis Italy has largely contributed: the functional calculus. Pincherle created the calculus of distributive operations, today called

linear functionals, and in his work we find many of the concepts and terms today accepted in the theory;[9] Giorgi particularly developed the symbolic calculus of Heaviside and gave valuable application of it to electrodynamical problems: he has reported on his contributions at the Toronto Congress of Mathematicians (1924).

But the functional calculus as a creative branch of analysis arises with the work of Volterra. From physical problems he drew the idea of considering functions depending not only on the values of one or more parameters, but on other functions, that is, in geometrical terminology not functions of one or more points, but functions of lines, surfaces and so on; hence the name of functions of lines replaced more recently by that of functionals.

It is impossible to refer in detail to the work of Volterra in this field, of which excellent expositions exist by Volterra himself and his school (Pérès, P. Lévy, Fantappiè). It will suffice to remember the integral and integro-differential equations which constitute important classes of functional equations, and to associate with the name of Volterra those of Fredholm, Hilbert, Picard, and to remember the more general studies of E. H. Moore and Fréchet on general analysis; and the list is necessarily too incomplete to give an idea of the enormous influence of the work of Volterra on all contemporary mathematical production.

These studies of such a general and abstract character found their most concrete application in the physical field, from which they were born, and in mathematical economics. Volterra himself gave the mathematical theory of hereditary phenomena in which the present state of a system depends not only on the present circumstances but on all its history.

Recently Volterra illustrated his theory of hereditary phenomena under considerations of energy: he showed that the work of external forces necessary to bring a system from a given state to a different state is always greater than the variation of a certain functional which depends exclusively on the present state of the system; and calculated the work dissipated by the external forces when a system returns to the initial condition.[10]

To these considerations Volterra was led by his recent researches on mathematical biology, whose consequences have been verified by the exploration of the seas.[11]

To illustrate another field in which integral and integro-differential equations show ample possibilities for practical use, I shall recall that Evans, a pupil of Volterra, and Roos have initiated with these means the study of economical phenomena in the regime of monopoly, and that F. P. Cantelli has used them largely in questions of calculus of probabilities and mathematical statistics. I shall say incidentally that an excellent treatise on the calculus of probabilities is that of Castelnuovo[12] who also contributed to the critical revision of its foundations.

[9] See Pincherle and Amaldi, "Le operazioni distributive" (Zanichelli, Bologna).

[10] See Atti del Congresso Internazionale dei Matematici a Bologna, 1928.

[11] See an article in "Scientia" (Bologna, Zanichelli, 1926).

[12] See the 2nd edition, two volumes, (Zanichelli, Bologna).

The researches of Volterra upon functionals relate generally to the real field, while those of Pincherle on distributive operations extend to the complex field. In the last few years Fantappiè has amply studied one class of functionals, the analytical functionals, which include the linear type of Pincherle as a particular case and which are analogous to the analytic functions. The introduction of the complex variable allows the coordination in a harmonic whole of the properties of this class of functionals. Fantappiè showed that it is possible to construct for such functionals a theory close to that of analytic functions, to give the analogues of Cauchy's Theorem and its series developments (more general then those of Volterra and Fréchet), and to study the polydromy of these functionals.

Lastly Giorgi and Fantappiè have fathomed the connections between functional analysis and the wave-mechanics.

Another branch of analysis is concerned with problems of the functional calculus: the calculus of variations whose first general treatment is due to Lagrange. Soon after his first essays Volterra had established the dependence of the calculus of variations on functional analysis. In fact, to the problems of maximum and minimum of a function correspond problems of maximum and minimum of a functional which constitute the very subject of the calculus of variations. The idea pointed out by Volterra was taken up again by Arzelà, who could not fully succeed because the functionals which presented themselves in the calculus of variations are not generally continuous. Tonelli, a pupil of Arzelà and Pincherle, was on the other hand completely successful. The Tonelli method is essentially based on the semicontinuity of such functionals, a concept analogous to that of semicontinuous functions given by Baire.

While the classical theories connect the study of the problems of variations with the differential equations of Euler, the Tonelli theory is independent of the theory of differential equations; on the contrary, the existence of the extremals satisfying given conditions is a consequence of the existence of the extremants which is demonstrated by direct proof.[13]

In the classical trend of Lagrange are to be mentioned the contributions of E. E. Levi, who gave sufficient conditions for an extremum, not depending on the knowledge of any fields of extremals, and a very simple proof of the necessity of Weierstrass's condition for the extremum of double integrals; of Burnengo who won analogous results for isoperimetrical problems; of Picone who has given new proofs, which hold also for the calculus of variations in two variables, of the Legendre and Jacobi conditions and a new suitable expression of the third variation.

Connected with the problems of the calculus of variations in many dimensions are the researches of Tonelli on the area of surfaces and the concepts of functions of two real variables absolutely continuous and with limited variations; concepts different from those already given by Vitali and Lebesgue. These concepts showed their utility in the theory of double Fourier's series, for the con-

[13] L. Tonelli, "Fundamenti di Calcolo delle Variazioni" (Zanichelli, Bologna).

vergence and uniform convergence of which Tonelli has given much simpler criteria than those already known.[14]

Lastly Caccioppoli, Nalli and Andreoli have introduced the new concept of an n-uple of functions of n variables with limited variation and of multiple integrals of Stieltjes which seem to have a great importance not only for the aforementioned questions but also for the definition of the integrals of functions of several complex variables. With a suitable extension of the concept of a function of many variables with limited variation, Caccioppoli has succeeded very recently in giving a simple analytical representation of multi-linear functionals of any degree, depending on an arbitrary number of functions (a problem which was treated by Fréchet in the case of bilinear functionals).

Let us now take a glance at the development of geometry in Italy, the country which the German geometer Klein has named the natural fatherland of geometry.

At the beginning of the 19th century arose in France, by the work of Poncelet (1821) the projective geometry, whose origin must be traced to the rules of perspective of the Italian artists of the Rinascimento.

With the rapid development of this branch of geometry in France, in Germany and in England, Italy was soon associated by the merit of Luigi Cremona-

While in the other countries of Europe this current of ideas seemed to wither with researches often sterile, in Italy it originated two branches, both very vital: the projective geometry of hyperspaces and the geometry of birational or Cremona transformations.

Projective geometry is nothing but the synthetic aspect of the invariant theory of algebraic forms with respect to the group of linear transformations; to this theory have contributed Betti, Beltrami, D'Ovidio, Bianchi, Capelli, Gerbaldi and more synthetically Bertini, Del Pezzo, Segre, Berzolari, Castelnuovo, Enriques, Severi, Scorza, Comessatti, Rosati, and many others. Some of the results of these researches have been collected in a book of Bertini, "Introduction to the projective theory of hyperspaces," of which a German translation by Duschek has recently been published. I will speak later of this branch of geometry in the differential field.

In the meantime Cremona was laying the foundations of the transformations which bear his name, which make correspond to lines higher curves (and not lines as in projective geometry). These transformations constitute a new instrument of geometrical analysis for the study of algebraic curves (or surfaces).

A very successful combination of the projective and the Cremonian trend, rich in further consequences, took place in the work of Segre and Castelnuovo, who transferred the properties which are invariant with respect to the birational transformations into projective properties of a convenient hyperspatial model; in this way projective geometry acquired a much larger domain and significance.

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[14] See his treatise on Fourier's series (Zanichelli, Bologna).

The studies of Riemann on abelian integrals, transferred by Brill and Noether to the algebraic field, found in Italy a particularly favourable ground and new methods of investigation. We may say that with the work of Bertini, Segre, Castelnuovo, Enriques and Severi the theory of rational functions of two variables connected by an algebraic equation, which means geometry on an algebraic curve, is substantially finished. To the theory of algebraic correspondences between algebraic curves and to their multiple relations with algebra and analysis, remarkable, and sometimes essential contributions, were given by Scorza, Rosati, Torelli and more recently Comessatti and Chisini.

The geometrical methods, already familiar to the Italian school, proved very successful when, about the year 1890, Castelnuovo and Enriques began to study the entirely new field of algebraic surfaces, which means the theory of rational functions of three variables connected by an algebraic equation.[15] At the very beginning of their researches they obtained a surprising result. When one tries to extend to surfaces the two definitions of the genus of a curve one finds two different numbers which are invariant under birational transformations of the surface: the geometrical genus and the arithmetical genus, generally different from each other. Thus was born the distinction of surfaces into two classes: the regular surfaces (for which the two genera are equal) and the irregular ones (with different genera) which manifest quite different properties with respect to the continuous systems of algebraic curves which they contain.

In the meantime Picard and Humbert in France extended to surfaces the theory of abelian integrals which was constructed for the algebraic curves by Riemann. We owe to Castelnuovo and Severi the conclusion of this period of researches with a definite result which connects the irregularity of a surface (that means the difference between its two genera) with the number of the simple distinct abelian integrals of the second species.

Severi constructed later the theory of the basis for algebraic curves on a surface, showing that the number of independent curves which constitute the basis is exactly the same as the invariant number introduced by Picard in the theory of the integral of the third species. To the theory of the basis for manifolds of many dimensions Comessatti and Albanese have also contributed.

Let us now go to the differential geometry.

With metric differential geometry are connected from the very beginning the names of Mainardi and Codazzi, who gave the fundamental equations of the Theory of Gauss, and of Beltrami, who also gave physical applications of it.

The most prominent representative of this branch of mathematics in Italy was Luigi Bianchi, whose influence was largely spread in Italy and abroad by his famous treatise.[16] The contributions of Bianchi and of his school[17] (E. E. Levi,

[15] See a report of G. Castelnuovo in the first volume of the Atti del Congresso Internazionale dei Matematici a Bologna, 1928 (Zanichelli, Bologna).

[16] "Lezioni di Geometria Differenziale," 3rd edition (Zanichelli, Bologna).

[17] See a report of G. Fubini in Annali di Matematica (1928–29) and of G. Scorza in the Annali della Scuola Normale Superiore di Pisa (1930).

Fubini, Picone, Calapso) are so large that it is even impossible to summarize them; I will only mention the theory of transformation of pseudospherical surfaces, and his famous theorem of permutability, the asymptotic transformations of surfaces, and the theory of deformations of quadrics; and these are only a few specimens of many wonderful results.

To the geometry of Riemann, Ricci gave a new and powerful impulse, whose value has been fully appreciated only after the creation of relativity theory, with the development of an algorithm which is named the absolute differential calculus or Ricci's calculus. This method, sometimes referred to as "mathematics of relativity," was largely used, in classical and relativistic mechanics, by Ricci, Levi-Civita, Amaldi, Palatini, and many others.

An essential contribution to the geometry of Riemann spaces was the notion of parallel displacement, introduced by Levi-Civita in 1917; the richness of ideas and results contained therein has been shown in Italy by Enea Bortolotti and myself, and outside of Italy by Schouten, Weyl, Eddington, Cartan, Wirtinger, Berwald and by the American School of which I shall name only the two leaders, Eisenhart and Veblen.

In this short review of metric differential geometry, in the sense of Gauss and Riemann, mention may be made of two different extensions of it given by Vitali and myself. I considered deformations of surfaces (or manifolds) in hyperspaces which conserve not only their linear elements, but also the curvatures, of a prefixed order, of their curves: the theory of these deformations of higher species is equivalent to the invariant properties of a system of differential forms of even orders. Vitali, with a remarkable extension of Ricci's calculus, was able to construct the analogues of Riemannian geometry for metric spaces of infinitely many dimensions, the Hilbert spaces.[18]

In the last decades of the past century and in the first years of our century, there was developing a new branch of differential geometry, which now constitutes a powerful chapter of our science: projective differential geometry.

Setting aside for a while the preceding and contemporary development of differential properties in hyperspaces, I will mention first Wilczynski, who between 1901 and 1910 gave a systematic treatment of differential geometry of curves and surfaces in ordinary space.[19]

The idea of representing curves, with respect to the projective group, by differential equations, belongs to Halphen (and was developed for hyperspaces by Berzolari); Wilczynski used in the projective representation of a surface in the ordinary space two partial differential equations (whose general integral depends, as in the case of Halphen, on arbitrary constants), which represent the existence of the asymptotic curves on it.

Fubini, beginning in 1915, has re-undertaken the study of differential projective geometry of surfaces in ordinary space using certain invariant differential forms (two quadratic and one cubic). The advantage of this method is two

[18] See his "Geometria degli-spazi hilbertiani" (Zanichelli, Bologna).

[19] Or projective space of three dimensions, S_3.

fold: first, it gives a representation of the surface which is invariant with respect to the aforementioned group (or also to the more general group of projective applicabilities, discovered by Fubini); second, it allows the use of Ricci's calculus, and therefore the immediate deduction of the invariants, covariants and contravariants of the surface.

The original deduction of Fubini's forms is rather complicated; I have succeeded in showing how from the equations of Wilczynski (which express a well known elementary fact) can be deduced the Fubini forms; and more than that I gave the geometrical meaning of these forms, until then unknown.

It is not possible to give an idea of the quantity of the new facts discovered in these last years in a field which we thought familiar and well known; however the treatise of Fubini and Cech,[20] where much of the theory is presented, exempts me from further reference.

A wider and higher field of research is offered by projective differential geometry of hyperspaces. The first researches in this field were made by Del Pezzo, who, about 1886, began to study synthetically the nature of the neighborhoods, of order higher than the first, of points on the surfaces and on manifolds in hyperspaces. From them he obtained remarkable results also in the algebraic field. These researches were practically forgotten when Segre, from 1906 to 1910, with suitable analytic representation, undertook the study of surfaces whose points have as coordinates solutions of a Laplace's equation and of manifolds generated by linear spaces.

In this new field, in which almost nothing was known and everything to be done, began to work Terracini and myself; a very accurate account of our achievements in the first of fifteen years of research has been given by Terracini,[21] so that I need not refer to particular results. It seems to me on the contrary important to point out some ideas which characterize the new trend with respect to the aforementioned researches.

In the study of surfaces in ordinary space the double system of asymptotic curves has a paramount importance; in fact a surface with such a double system necessarily belongs to a space of three dimensions; that is why the general integral of Wilczynski's equations depends only on arbitrary constants.

When one tries to extend these considerations to surfaces or manifolds in hyperspaces, the necessity appears immediately of finding such system of curves, projectively determined on the surface, whose existence characterizes the dimensionality of the space to which the surface belongs. In order to do that, I introduced the quasi-asymptotic curves, which are determined by suitable conditions of intersection of the osculating spaces (of a certain order r) to the curve and the osculating spaces (of a certain order s) to the surface (for the ordinary asymptotics $s=1, r=2$). This was the first necessary geometric step to construct a general theory of surfaces in a space of any finite dimensionality: the analytical expression of these conditions gives the differential equations satisfied by the

[20] "Lezioni di Geometria Proiettiva Differenziale" (Zanichelli, Bologna).

[21] See his Appendix III in the treatise of Fubini and Cech.

surface. The calculation of their integrability conditions, covariants and invariants is then a matter of course and of routine; better than that is to introduce (as we have pointed out for surfaces in S_3) some invariant forms which give an invariant representation of the surface and possibly allow the use of Ricci's Calculus. Another step, which requires again geometric skilfulness, is the geometric interpretation of such invariant forms.

The preceding considerations lead, by the very statement of the problem, to systems of partial linear differential equations whose general integral depends only on arbitrary constants (whose number is determined by the dimensionality of the space to which the surface is supposed to belong).

However, it is not always necessary to fix the dimensionality when dealing with a problem in partial differential equations; on the contrary in some questions it is necessary to leave it among the unknowns. In fact the first results I obtained in freeing myself from this dimensionality were the necessary and sufficient conditions, geometrically deduced, in order that a Laplace's equation could be integrated by quadratures; a problem proposed by Darboux and which, having resisted the efforts of Goursat, was still awaiting a solution.

From this example we arrive at the most general and fertile point of view reached in this branch of geometry; we do not consider necessarily a particular group of solutions of a system of linear partial differential equations in two variables, whose general integral depends on arbitrary constants as representing the points of a surface. Any set of functions (finite or infinite in number) satisfying or not satisfying a system of differential equations, whose general integral may also depend on arbitrary functions, represents points of a surface; the dimensionality of the space (which may also not be finite) is unessential (if not determined by the given system of equations). What is on the contrary essential is the dimensionality of the spaces which contain the neighborhoods of the different orders of the points of the surface, which I call the osculating spaces, and the fact that in these spaces (always of a finite dimensionality; and not necessarily in the ambient) the projective geometry still holds. And so we reach really a projective geometry of linear partial differential equations (and not of particular types of them or of some of their solutions) and more generally of any set of linearly independent functions.

For these surfaces (or manifolds), although so general, it is possible to give a theory of point to point correspondences, as I have outlined in a communication to the International Congress of Mathematicians in Bologna (1928); and it is also possible to give a constructive meaning to the geometry of paths and to theory of projective parallel displacement, as I have shown in some lectures given in American Universities.

I have thus come to the end of this review, too extended for you perhaps, but certainly too short for the wideness of the subject. I hope that I have succeeded in giving you, even through the defiiciency of my exposition, an idea of the part that Italy takes in this truly international collaboration for the progress of human knowledge.

QUESTIONS AND DISCUSSIONS

EDITED by R. E. GILMAN, Brown University, Providence, Rhode Island.

The Department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

In the November, 1930 issue of this Monthly, on page 493, there appeared "A simple proof of the theorem of Morley"[1] by Mr. Jacob O. Engelhardt. Since then the editor has received the following two letters which show in different ways how that proof can be simplified. Neither one of the modified proofs involves Mr. Engelhardt's Lemma 2.

I

Institut Henri Poincaré, Paris, le 30 Nov., 1930.

Monsieur et cher collègue:

La demonstration fort ingénieuse du théorème de Morley,[2] due à M. Jacob Engelhardt, me parait pouvois être encore simplifici comme il suit. Reprenous la figure de la p. 493 et posons

$$\angle BFD = \alpha, \quad \angle BDF = \gamma, \quad \angle CDE = \beta.$$

M. Engelhardt obtient

$$BD = 4 \sin x \sin z \sin (60° + x), \quad BF = 4 \sin x \sin z (60° + z).$$

On a donc

$$\frac{\sin \alpha}{\sin \gamma} = \frac{BD}{BF} = \frac{\sin (60° + x)}{\sin (60° + z)}.$$

D'autre part

$$\alpha + \gamma = 180° - y = 60° + x + 60° + z.$$

Par suite $\alpha = 60° + x$, $\gamma = 60° + z$. On montrerait de même que $\beta = 60° + y$. Par suite, en remarquant que $\angle BDC = 180° - y - z$.

$$\angle FDE = 360° - \beta - \gamma - \angle BDC = 60°.$$

Sentiments distingué,
ÉMILE BOREL.

II

Arizona State Teachers College, Flagstaff, Arizona, Dec. 1, 1930.

The proof of the theorem of Morley given by Mr. Jacob O. Engelhardt on page 493 of the Nov., 1930 issue of this Monthly can be simplified as follows:

He obtained the relation

$$(FD)^2 = 16 \sin^2 x \sin^2 z [\sin^2 (60° + x) + \sin^2 (60° + z) - 2 \sin (60° + x) \sin (60° + z) \cos y].$$

[1] The theorem of Morley is as follows: If three angles of a triangle be trisected, the triangle whose vertices are each the intersection of a pair of trisectors adjacent to a side is equilateral.

[2] This Monthly, vol. 37 (1930), p. 493.

The expression within the brackets is equal to $\sin^2 y$ by virtue of the law of cosines applied to the triangle whose angles[1] are y, $60°+x$, and $60°+z$ and whose circumscribed circle has a radius equal to $\frac{1}{2}$. Therefore

$$(FD)^2 = 16 \sin^2 x \sin^2 y \sin^2 z \text{ and } FD = 4 \sin x \sin y \sin z.$$

Since the expressions for DE and EF are obtainable from the expressions for FD by permuting x, y, and z, and since FD is a symmetric function of x, y, and z, it follows that $FD = DE = EF$.

W. C. RISSELMAN

MATHEMATICS CLUBS

All reports of club activities should be sent to Professor F. M. Weida, George Washington, University, Washington, D. C.

CLUB ACTIVITIES

The Mathematics Club of George Washington University, Washington, D. C.

The officers for the year 1929–1930 were: President, Dr. F. E. Johnston; Secretary, Mr. Michael Goldberg.

The programs for the year 1929-1930 were as follows:

October 21, 1929. "The packing of spheres and hyperspheres" by Mr. Michael Goldberg.
November 4. "A generalization of Riemannian geometry" by Dr. James H. Taylor.
November 18. "Divergent series" by Dr. Florence M. Mears.
December 2. "The evolution of the concept of infinity" by Dr. Tobias Dantzig.
December 16. "The geometry of rigid dynamics" by Dr. Paul Wernicke.
February 17, 1930. "Modern hydromechanics" by Dr. Edgar W. Woolard.
March 3. "Alignment charts" by Mr. Albert Wertheimer.
March 24. "Order of contact of curves" by Mr. Paul Bradt.
April 7. "The summation of series" by Mr. William J. Berry.
May 5. "Primitive roots of prime numbers" by Dr. F. E. Johnston.
June 14. Picnic at Chapel Point, Md.

(Report by Michael Goldberg)

The University of Louisville Mathematics Club, Louisville, Kentucky.

The University of Louisville Mathematics Club was organized in October, 1929. We feel that we have accomplished a great deal in creating during the first year of our existence "a greater interest in and knowledge of mathematics," which is the aim of our organization. We owe our excellent beginning to our advisers, Dr. Guy Stevenson and Dr. Walter Moore. We have an enrollment of twenty, the membership of the club being restricted to persons interested in mathematics.

The officers of the club for the year 1929–1930 were: President, Arthur Ries ('31); Vice-President, Virginia Lee Brightwell ('30); Secretary, Dorothy Fleischmann ('30); Treasurer, Helen Kline ('31); Sergeant-at-Arms, James Teller ('32).

The following programs were given at the regular bi-weekly meetings:

October 30, 1929. Election of officers.
November 15. "The art of paper cutting" by Dr. Moore.

The club was entertained by Dr. and Mrs. Moore.

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[1] In this triangle the sides opposite these angles are respectively $\sin y$, $\sin(60°+x)$, $\sin(60°+z)$.

December 5. "Hyperbolic functions" by Arthur Ries.
December 19. "Mentally extracting the cube roots of numbers from one to one billion" by Dr. Sappenfield.
January 9, 1930. "Mathematical treatment of honey cells" by James Teller; Election of officers.
January 25. "Number theory" by Dr. Stevenson.
February 6. "Magic squares" by Mariam Heymann; Initiation of four new members.
February 20. "Squaring the circle" by Grace Lothman.
March 6. "The transcendence of π and e" by Dorothy Fleischmann.
March 20. "The nine point circle" by Wallace Denhard.
April 3. "Apollonius's problem" by Ruth Vogel.
April 16. "The history of mathematics" by Virginia Lee Brightwell; Picnic in Cherokee Park.
May 1. "Mathematical tricks" by George Chenault. The club was entertained by Dr. and Mrs. Stevenson.
May 15. "Number systems" by Edwin Bell; Election of officers. The club was entertained by Dr. and Mrs. Sappenfield.
May 29. Installation of officers; Picnic at Iroquois Park.

(Report by Dorothy Fleischmann)

The Mathematics Club of Hunter College of the City of New York.

The Mathematics Club of the Junior-Senior years at Hunter College held meetings at two-weekly intervals throughout the year 1929–1930. The programs of the meetings included popular topics, such as the trisection of an angle by various methods with a proof of the impossibility of the construction by means of straight-edge and compasses; the golden section and the construction of certain regular polygons; the geometric construction of a regular seventeen-sided polygon and the construction from an algebraic point of view; Non-Euclidean geometry; the fourth dimension; the construction of a straight line by means of the Peaucellier inversor. An especially snappy talk was given by the secretary of the Club on "Geometry of the compasses;" some of the members who were doing honors work in mathematics gave a discussion of "The Dedekind cut," and another member gave a talk on "The history of permutations and combinations" based on an intensive study made in connection with an advanced course in the history of mathematics.

The Club had the privilege of hearing two outside speakers. Professor W. S. Schlauch of New York University gave an address on "Mathematics as an interpreter of life," and Professor Jean Conklin, formerly of the College, gave a very timely talk on "The new planet X."

The outstanding social event of the year was the presentation of a play by the students of the Mathematics Club of the Brooklyn Branch (now a part of the Brooklyn College). It was entitled "Interlopers" and was the adaptation by a member of the faculty of a play by Professor Slaught which appeared in the March, 1928 issue of this Monthly as "The Evolution of Numbers." The large audience was fully rewarded by the lively entertainment which was provided. Besides this event, there was a social affair each semester; in the fall a play "Alice in the Wonderland of Mathematics" was given; in the spring, a feature of interest was a series of songs written by another member of the faculty. The one that proved to be the most popular was sung to the tune of "The Battle Hymn of the Republic" with a ringing chorus:

"Glory, glory to our major,
Glory, glory to our major,
Glory, glory to our major,
Its truth is marching on."

A happy day, June 16th, spent on a trip up the Hudson ended the social activities of the year.

Both the president and secretary are spending 1930–1931 at Bryn Mawr College on graduate scholarships. The officers were: President, Miriam Fassler; Vice-president, Rebecca Rosenblum; Secretary, Julia Cincotti; Treasurer, Violet Moskowitz; Publicity Manager, Muriel Rosner; Faculty Adviser, Prof. Evelyn Walker.

(Report by Lao G. Simons)

The Mathematics Club of the University of Kansas.

Officers: President, Ella Baker; Vice-President, Velt Stafford; Secretary-Treasurer, Helen Kemp; Faculty Adviser, Professor Florence Black; Social Chairmen, Ruth Pratt, Maurice Brown.

The programs for the year 1929–1930 were as follows:

October 14, 1929. Business Meeting.

October 28. Old fashioned ciphering match; refreshments.

November 11. "The left hand of learning" by Professor U. G. Mitchell.

November 25. Methods of solving equations by approximations: (1) "Newton's method" by Ella Baker; (2) "Interpolation" by Philip Bell.

December 9. (3) "Iteration" by Ralph Sickel; (4) "General discussion of the first three" by Professor Jordan.

January 13, 1930. "Interesting problems brought to the mathematics department" by Professor J. J. Wheeler.

February 10. "Div-a-lets" by Vida Dunbar.

February 24. "Magic squares" by Lenore Cummings.

March 10. "Mathematics as applied to artillery work" by Lieutenant Harry F. Myers.

March 24. "Special relativity" by Professor Stranathan.

April 14. Business meeting; Election of Officers.

April 28. "Methods of rapid calculation in business" by Mr. Bell.

May 14. Annual mathematics club picnic. (Report by Florence Black)

The Ohio Wesleyan University Chapter of Pi Mu Epsilon.

The Beta Chapter of Ohio of Pi Mu Epsilon at Ohio Wesleyan University, Delaware, Ohio held regular monthly meetings during the college year 1929–1930. Some of the subjects discussed were: "Condensers," "Plücker coordinates," and "Polar planimeters." Dr. D. C. Miller of the Case School of Applied Science, gave a very interesting lecture in May on the subject, "Sound."

(Report by Raymond F. Felts)

Pi Mu Epsilon and the White Mathematics Club of the University of Kentucky, Lexington, Kentucky.

Meetings of the two organizations alternate. The combined program for the year was the following:

November 20, 1929. "On cubic congruences" by Professor C. G. Latimer.

December 12. "Pot-pourri of mathematics" by the faculty. A meeting for freshmen.

December 19. "On interpolation in mathematics of finance" by Professor D. E. South.

January 9, 1930. "Modification of a proof by Steiner" by Mr. E. J. Canaday.

January 23. "Functions of limited variation" by Professor H. H. Downing.

February 13. "Queen Dido's problem" by Professor Elizabeth Le Stourgeon.

March 6. "Linkages" by Miss Sallie Pence.

March 20. Initiation of new members into Pi Mu Epsilon.

March 27. "On certain graphic considerations" by Professor M. C. Brown.

April 24. "De Jonquière's theory of indices" by Dean P. P. Boyd.

May 1. "Graphic construction of the roots of a quadratic equation" by Miss Alleen Lemons. "Classification of quadric surfaces by invariants and covariants" by Mr. N. B. Allison.

May 15. "An identity in theta functions" by Mr. Smith Park. In the evening the local chapter of Pi Mu Epsilon was entertained at the home of Dean P. P. Boyd. The festivities included initiation of new members.

(Report by F. Elizabeth Le Stourgeon)

The Mathematics Club of Oberlin College, Oberlin, Ohio.

The Mathematics Club of Oberlin College, founded in 1894, was reorganized after a lapse of time in 1922. Since that date, meetings have been held every two weeks of each college year.

Papers are given by students majoring in mathematics, the privilege being extended occasionally to a gifted freshman. Faculty papers are rare, but greatly appreciated. Candidates for a master's degree and undergraduate students who are candidates for honors usually present reports connected with their reading or research. Attendance at the meetings averages about thirty. The climax of the year is the annual May banquet, at which the club is honored by the presence of some mathematician of distinction. Attendance at last year's banquet was forty-five.

The officers for 1929–30 were: President, Ernest Peek ('30); Vice President, Dorothea Zilch ('30); Secretary-Treasurer, Gertrude Brockett ('30); Social Chairman, Frances Gifford ('30). The program committee consisted of Gertrude Brockett, Eugene Buell, and Professor Mary E. Sinclair, faculty adviser for the club.

The programs for the year 1929–1930 were as follows:

October 18, 1929. "Highest common factor" by Dorothea Zilch; "Related theorems about numbers" by Ernest Peek; "Problems suggested for solution" by Professor Marie Johnson.

November 1. "Families of curves of pursuit" by Emily Grace Doane; "Suggested readings in mathematics" by Miss Sinclair.

November 15. "Number domains" by Eugene Buell; "Irrational numbers" by Frances Gifford.

November 29. "Geometric constructions with compasses alone" by Talbot Harding; "Properties of digits of numbers" by Ruth Brummitt.

December 13. Christmas Party. Presentation of the mathematical play "The Evolution of Numbers" by Professor Slaught.

January 10, 1930. "Mathematics as a mode of thought" by Willard V. Quine; "Analogies in plane and spherical trigonometry" by John Insprucker.

January 17. "Large modern telescopes" by John Dudley; "Mathematical prodigies" by Evelyn Smith.

February 7. "The gyroscope" by Professor Carr,; "Congruences, modulo m" by Joseph W. Nadal.

February 21. "Rabbi Ben Ezra and the Hindu-Arabic numerals" by Faith Fitch; "Problems in maxima and minima" by Robert Lemmerman.

March 7. "Problems connected with central forces" by Ross Wilson; "Expression by series of trigonometric, hyperbolic, and exponential functions" by Ruth Roudabush.

March 21. "Actuarial mathematics" by Mary Elizabeth Schubert; "The Pythagorean theorem" by Dorothy Rainer.

April 11. "Curves related to the equilateral hyperbola" by John Hardy; Honors report by Frederick Ficken.

April 25. "Charts and graphs" by Robert Wilkins; Honors report by Gertrude Brockett.

May 7. "Theory of logarithms" by Helen Winder; "Mathematics and music" by Margaret Draeger.

May 23. Banquet. Address by Professor G. A. Bliss of the University of Chicago.

(Report by Gertrude Brockett)

CLUB TOPICS

1931 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS

By WALTER CROSBY EELLS, Stanford University, California.

In continuation of previously published lists[1] of centennial dates in the history of mathematics, the following list of important 1931 centennial dates is presented.

B.C. 570[2] Birth of Anaximenes, prominent pupil of Thales, of Ionian school, Greek mathematician and astronomer.

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[1] This Monthly, vol. 37 (1930) pp. 150–151 for a list of 1930 events and for references to previous volumes for corresponding lists from 1925 to 1929.

[2] In lists of dates, similar to the present one, which I have furnished this Monthly since 1925, occasionally events have been included which occurred prior to the beginning of the Christian era. For example in the 1930 list, 570 B.C. was given for the birth of Anaximenes, which is repeated in

B.C. 470 Birth of Hippocrates of Chios, one of the greatest Greek geometers, celebrated for his study of the quadrature of lunes.

370 Death of Democritus of Abdera, Thracian geometer.

70 Geminus of Rhodes, Greek historian of mathematics, flourished.

A.D. 1631. Publication of William Oughtred's *Clavis Mathematicae*, systematic textbook on arithmetic containing practically all known on the subject. Introduction in it of the symbol $\times$ for multiplication.

1631. Thomas Harriot, English mathematician, introduces symbols $<$ and $>$ for "is less than" and "is greater than."

1631. Death of Briggs, who was largely responsible for the invention, computation, and popularization of common logarithms.

1731. Birth of Cavendish, mathematical physicist, who determined the density of the earth by comparison with two lead balls.

1731. Publication of Alexis Clairaut's *Recherches sur les courbes a double courbure*, which secured his admission to the Paris Academy of Sciences when still under legal age.

1731. Death of Brook Taylor, discoverer of Taylor's theorem for the expansion of $f(x+h)$.

1831. Birth of Richard Dedekind, German mathematician celebrated for his work on number theory.

1831. Birth of Amedee Mannheim who designed the Mannheim slide rule.

1831. Birth of James Clerk Maxwell who revolutionized electro-magnetism and established mathematically the electro-magnetic theory of light.

1831. Birth of P. G. Tait, professor at the University of Edinburgh, who formulated the law of the flight of the golf ball after one of his sons had become a brilliant golfer, and who made important discoveries in physics and quaternions.

1831. Discovery by the French mathematician Cauchy of the general theorem which reveals the number of roots, real or complex, which lie within a given contour.

1831. Publication of Plücker's *Analytisch-Geometrische Entwicklungen*, containing improved abbreviated notation.

1831. Death of Sophie Germain, Parisian mathematician, best known for her work on the theory of elastic surfaces.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York.

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Mathematical Introduction to Economics. By Griffith C. Evans. New York, McGraw-Hill Book Co., 1930. xi+177 pages. $3.00

Many writers on economic theory have used mathematical ideas, a few have

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this year's list. Mr. Edward S. Smith in "A Letter to the Editor" (This Monthly, vol. 37, 1930, pp. 371–72) points out that I have been in error by one year with reference to all dates B.C. He is entirely right and I am very glad to be corrected. The error, which is easily made, is due to the fact that the first year of the Christian era was number *one*, not *zero*. The civilized world has almost universally made the same error in celebrating the bimillenium of Virgil's birth (Oct. 15, 70 B.C.) in 1930 instead of in 1931. See a note by the author, "Virgil Bimillennium Celebrated Prematurely" in *School and Society*, vol. 32 (December 13, 1930), pp. 805–806.

used mathematical notation, and a much smaller number have used mathematics more advanced than the most elementary parts of calculus. Of the last group, Professor Evans was the earliest, with an article in this Monthly[1] for February, 1924, on "The Dynamics of Monopoly," using the calculus of variations; at least he was the earliest if we except certain allusions to the higher mathematics required by statistics, H. L. Moore's work with Fourier series, and the mention in Irving Fisher's doctoral thesis of integrability conditions. However Professor Evans is now writing for those who know no more than calculus. Differential equations and the calculus of variations are used, but the reader is not frightened away by these formidable names, for they do not appear in the book until the fundamental ideas back of them have been insinuated into the mind in the form of economic problems.

Such classical problems as the bartering of nuts for apples give way in this book to questions arising more directly in a manufacturing age. Instead of beginning with supply and demand functions, the first chapter, whose general subject is monopoly, begins with cost and demand functions. Exchange between two individuals and utility are dealt with in the twelfth and eleventh chapters.

Chapter 2 brings out the principle that economic laws should be in forms invariant under changes of units. Chapter 3 deals with competition and what the author terms "cooperation" among a finite number of producers. In Chapter 4 offer and demand depend upon the rate of change of the price as well as upon the price itself; in the final section demand depends upon a sum or integral of past prices. Enter, *sine nomine*, differential and integral equations. Chapter 5 deals with changes in cost and demand functions, and particularly with those changes caused by taxation, while diversified costs, tariffs, and rent form the subject of the next chapter. Further chapters treat of foreign exchange, interest, the equation of exchange, price indices, business cycles, general concepts, production. Finally there are two chapters of more advanced material, with the calculus of variations entering at first anonymously and in a very special problem; a good bibliography of mathematical economics, and an appendix discussing certain further ideas such as the "compartment" theory of economics which Professor Evans set forth in the Proceedings of the National Academy of Sciences in 1925.

When the quantity purchased is written as a function of all past prices as well as of the current one, would it not be well to consider also situations in which the price which buyers are willing to pay is a function of past and current quantities consumed? Indeed the latter situation seems the more widespread, in view of the persistence of habits, particularly food habits, and the inertia produced by the existence of capital equipment involved in utilization. The population of China, though starving, buys immensely more rice than wheat even when the rice costs $2\frac{1}{2}$ times as much per calorie.

In dealing with n competitors the text follows tradition in assuming that

[1] Vol. 31 (1924), pp. 77–83.

all sell at the same price, though they produce different quantities. It would be illuminating to bring out the duality of price with quantity by writing the n prices as functions of the n quantities with non-vanishing Jacobian. This same scheme could be used in treating correlated demand and joint costs. The idea that there must be one single price for all sellers, of whatever sort and wherever located, merely because their products go by the same generic name, is untenable. A little considered but very important feature of competition has been brought out elsewhere by the reviewer: with reference to each seller there are groups of buyers who will deal with him in preference to others, in spite of a moderate price difference, which may vary continuously among buyers. Transportation costs and many other factors varying among buyers supplement price in affecting their decisions. The usual conception of a market as a point, in which there can be only one price at a time, needs to be extended to the notion of a market as something like an area. The result will be a more elegant as well as a more realistic theory, since prices will enter in a manner symmetrical with quantities, and will have a vital bearing on the stability of a competitive situation.

Edgeworth's paradoxical discovery that a tax on an article controlled by a monopolist who also controls a substitute commodity may lead him to reduce both prices, besides paying the tax, might very well be brought out in a course in mathematical economics. This is one proposition of economic importance which everyone can appreciate but which cannot be proved, apparently, without the use of formulae; moreover it is very suggestive of further developments.

Will this or some similar book ever come to replace the bulky "Elements of Economics" now used as sophomore texts? Not, certainly, for the majority of those who now peruse these wordy treatises. But can Professor Evans' book replace such works in the liberal education of the intelligent mathematician, who is bored even as a sophomore by Economics 1 and demands stronger meat? Perhaps a compromise will be worked out. Perhaps some day we shall have university courses in "Economics for Mathematicians," as well as in "Mathematics for Economists."

Any university could immensely improve the average quality of its graduates in economics by requiring that they must all master Professor Evans' book, solving simultaneously the difficulty about overcrowding. This simple and effective remedy for two evils will hardly be taken seriously at the present time. But it is clear that this book and other recent work in mathematical economics by Roos, Schultz, Moore and others will help to bring nearer the time when calculus will be considered an indispensable part of a liberal education.

HAROLD HOTELLING

Algebraic Equations. An introduction to the Theories of Lagrange and Galois. By Edgar Dehn. Columbia University Press, 1930. xi+200 pages. $4.25.

A number of modern expositions of the theory of equations from the group stand-point have given the reviewer the impression that their authors essayed

to present that elegant theory with a minimum number of words, with the result that the student who approaches the subject for the first time soon finds himself lost in a maze of new concepts from which he may extricate himself if he has sufficient time and pugnacity and knows how to devise illustrations which will fix these concepts in mind. The reader who has had this impression will welcome Dr. Dehn's book because of the slowness with which the theory is developed, the attention which is paid to details, the repetition of difficult and important concepts, and the care which the author takes not merely to prove theorems but to give the student a perspective on the subject as a whole.

No less than five chapters are devoted to the theory of permutations and permutation groups in order that the student may properly cultivate the technique necessary for a comprehension of the theory. If the author had only included a list of exercises, little more could have been asked. Although Lagrange's theory is included in Galois' theory, we are glad to observe that the author has devoted two chapters to Lagrange's work, explaining the weaknesses of Lagrange's theory in detail without diminishing the tribute which he pays to Lagrange's genius. The reviewer agrees with the author that a knowledge of Lagrange's theory is necessary for a proper appreciation of Galois' theory.

We regret to observe that the text is not free from defects. On page 24 the author states in passing that the relation

$$(stu \cdots)^2 = s^2t^2u^2 \cdots$$

implies that every pair of the substitutions $s, t, u, \cdots$, must be commutative. This is not true, as is shown by the example: $s=(123)$, $t=(12)$, $u=(132)$.

On page 112, in considering a lemma which is useful in proving the Jordan-Holder theorem, the author states that if N and N' are maximal invariant subgroups of G, then $G=NN'$, implying by this notation that every element of G is expressible as a product of an element of N by an element of N'. This should be proved.

A more serious gap occurs on page 170 where the author proves that, if the group of an equation is reduced by an adjunction to the domain of rationality, the same reduction can be effected by adjoining a root of a resolvent equation, and infers that a solvable equation has a solvable group. The gap may be bridged by proving that the group of the binomial equation $x^n=a$, in a field containing all the nth roots of unity, is a cyclic group, and then invoking the theorems of §75.

The student who expects to consult other works on the same subject must be warned that Dr. Dehn's definitions of certain technical terms differ from those of other writers. The algebraic operations, according to Dr. Dehn, include the rational operations and the extractions of roots; most algebraists consider the process of solving an algebraic equation or of adjoining a root of an equation to a field an algebraic operation. Consistent with this definition is his definition (page 131) of a relative algebraic domain, which conflicts with the

usage of this phrase by writers on algebraic numbers. The phrase "class of permutations (page 118) is translated literally from the German; we wish Dr. Dehn had used the recognized English equivalent "complete set of conjugates."

We note that the author defines (page 27) a permutation group as a set of permutations which leave a function unaltered, and then reconciles this definition with the more familiar one. Consistent with this definition he employs the idea of functions belonging to a group in considering Lagrange's theorem (page 30) and conjugate groups (page 33). This procedure may be defensible on pedagogical grounds, but many will object on the general principle that a proof of a theorem should, if possible, involve only essential concepts.

LOUIS WEISNER

Neuere Methode und Ergebnisse in der Hydrodynamik (Band I, *Mathematik und ihre Anwendungen in Monographien und Lehrbüchern*). By C. W. Oseen. E. Hilb, Leipzig, 1927. Akademische Verlagsgesellschaft M.B.H.

In this volume Oseen assembles the researches on fluid dynamics which date from the year 1910. No small part of these important results are due to Oseen and his pupils, Faxén and Zeilon, and published in Scandinavian journals. The results in the main concern motions of viscous fluids together with the limiting case of zero viscosity. It is found for example, that the results in this limiting case differ widely from the theory of perfect fluids but are more in agreement with the facts of experiment and observation, than the theoretical results based on ideal frictionless fluids.

The author first undertakes to investigate the linear system of partial differential equations which are obtained by neglecting the second degree terms in the general equations of motion of a viscous but incompressible fluid. These equations are valid for slow motions and are important for the field of colloidal chemistry. By this means the author resolves Whitehead's paradox. Lamb has also resolved Stokes' paradox, by Oseen's methods.

The ultimate objective of the author was to pass in complete fashion from the case of viscosity, to the limiting case of zero viscosity. This end has not been reached for the general system. For the linear case, the passage to the limit has been achieved. The results thus obtained are substantially in agreement with the results for actual fluids.

It appears then that the so-called paradox of D'Alembert that in the steady motion of a perfect fluid there is no pressure exerted against an immersed body, rests on a faulty passage to the limit. Zeilon showed more than this negative result. In Part III his results describe quantitatively the resistance and pressure on the head end of the immersed body.

In the first chapter of the present work, the hydrodynamic differential equations are derived, first on the assumption that the first derivatives of the pressure function and velocity components with respect to the position coordinates, and the accelerations, are continuous. The impulse theorem is used. Later the fundamental system of equations is derived in the form of integro-differential

equations directly from mechanical principles and assuming merely integrability of the above quantities. At the end of the first section, the special cases of steady motion with respect to a fixed reference system and motion with uniform velocity are discussed.

In the second section, the method of solution to be used later is illustrated by means of the generalized Poisson differential equation. For the interior problem a power series solution in a parameter is obtained from an infinite system of Poisson equations. Green's theorem and Green's functions are used. For the exterior problem a characteristic difficulty is exhibited arising from the fact that the power series is not convergent in the neighborhood of the zero value of the parameter.

In the third section, a generalized Green's theorem is used for the Stokes form of the system of differential equations (linear approximation). The fundamental solutions are found and by means of these the complete set of equations for steady motion are transformed into integro-differential equations.

In the fourth section, the generalized Stokes equation is solved and the fifth section gives the solution of the most general linear system.

In section seven, the first application of the integro-differential equations is made. The problem is that of determining the motion of a viscous but incompressible fluid, filling all space, when the initial velocity is known. The following result is obtained: if the motion is regular (defined in a certain specific sense) at $t=T$, then there exists a positive number $\tau>0$ such that it is regular in the time interval $T<t<T+\tau$. However, the motion may cease to be regular and become turbulent at some later time.

Section eight, the last of Part I, deals with the simplest examples of turbulence in a viscous fluid. The two problems considered are (1) a single straight line vortex with rotational symmetry and (2) the interaction between two such line vortices. The solution for the first problem is obtained in exact form immediately from the integro-differential equations, and is only a little more complicated than the Helmholtz solution for the ideal fluid.

Part II treats the boundary value problems proper of hydrodynamics. In section 9, an exact solution is given of the linear Stokes equation for steady motion subject to the conditions that the velocity vanishes at infinity and takes on prescribed values on the surface of a sphere. Lamp had already obtained a series solution, but here the solution is given in closed form. For special cases, more convenient and direct methods can be used than employing the general solution. The Stokes formula for the pressure on a sphere, moving with constant speed in a fluid, can be obtained, such as raindrops through the air, fine particles suspended in a liquid, etc. At the end of section 9, the solution for the Stokes equation subject to prescribed boundary values on a given plane is obtained.

Section 10 yields an approach to the general hydrodynamical problem of non-steady motions. From the results obtained at this point, a solution of the Prob-

lem of Boussinesq is obtained on the non-steady motion of a small sphere in a viscous fluid.

Section 11 reproduces the theory of the steady motion of an ellipsoid in a viscous fluid due to Overbeck. This is given in preparation for later work.

Section 11 points out that even though the use of the Stokes equations cannot be justified in general, the results for the motion of a sphere for a sufficiently small neighborhood of the sphere are in substantial agreement with the data of colloidal chemistry.

Section 12 is a summary of the investigation of H. A. Lorentz on the steady motion of a sphere in a viscous fluid which is bounded by a plane wall.

Section 13 investigates the motion of a sphere between two parallel plane walls.

Section 14 is a report on the vortex interaction of two spheres moving in a viscous fluid. The results are due to Smoluchowski, Faxén, Dahl, Stimson, and Jeffery.

Section 15 points out that the paradoxes of Stokes and Whitehead are resolved if we note as in Section 11 that the Stokes equations are not approximations for the general equations.

The extended or generalized Stokes equations are obtained by Oseen by retaining those terms of the general system which should not be neglected. These equations and the accompanying boundary value problems are solved in the second half of part II.

Section 16 solves the steady motion of a sphere for a viscous fluid by means of the extended Stokes equation. The solution thus obtained differs in essential character from that of Stokes. The chief difference is that while Stokes' solution has fore and aft symmetry the solution of Oseen has decided d ssymmetry in the form of a tail of vortices behind the sphere. A new term is obtained as a second approximation proportional to the square of the velocity. In addition a second approximation is obtained for the motion near the sphere.

Section 17 deals with the motion of a circular cylinder related to the work of Lamb, Barstow, Cone, and Lang.

In Section 18, the ellipsoid is considered and Overbeck's results appear as a first approximation. In the second approximation terms appear in the expression for pressure proportional to the square of the velocity. For the elliptic cylinder results due to Harrison are stated without proof.

Section 19 contains Faxén's extensions of Lorentz's work given in section 12 and in section 20, Faxén's extension of Ladenburg's work on motion of a sphere in a tube is given.

Section 21 treats the same problem as in Section 14 from the point of view of the extended Stokes' equation.

Part III contains by far the most important results, viz: the passage to the limit of zero viscosity, carrying over the solution for the extended Stokes equation.

Section 23 carries this through for a thin plate. In Section 24 this is extended

to a more general surface. In Section 25, the same problem is carried through for a solid body.

In Section 26, the solutions of certain Fredholm integral equations are used to obtain new results for the problem of steady motion of solid bodies.

Sections 27–30 make use of a method due to Hilbert to solve the steady motion of a general cylinder, circular cylinder and flat plate. These results are of fundamental importance in the computation of the lift and drag forces of aeronautics.

Finally in Section 31 by means of elliptic functions the motion of a circular plate is developed.

In addition to the valuable results themselves it is of interest to realize the even greater use which is being made in various branches of mathematical physics of boundary value problems in ordinary and partial differential equations.

H. J. ETTLINGER

Intermediate Mechanics. By D. Humphrey. Longmans, Green and Company, London and New York, 1930. $4.20.

Applied Mechanics. By Norman C. Riggs. The Macmillan Company, New York, 1930. $3.75.

These two books, although written for apparently different purposes, have many common points of merit, so that they may be discussed together. While "Applied Mechanics" is designed primarily for students of engineering, particularly those who are interested only in the technical aspects of engineering, it should make an especial appeal to that relatively small group of engineers who desire to extend their knowledge beyond the mere "rule of thumb" into the deeper foundations and developments of the general theory. On the other hand, "Intermediate Mechanics," intended for students of physics, might also be used to advantage by the engineers.

Both books are elementary in the sense that they begin by developing the concepts and postulates peculiar to mechanics. The treatment of vectors in "Applied Mechanics" requires a knowledge of geometry alone. Velocity, acceleration, and force require the notion of a limit and, in the case of these books, it is so well presented that the student should have but little difficulty in understanding them. They are particularly attractive from the fact that clear and precise definitions are formulated mathematically whereby applications can be made directly to the problems.

Such topics as D'Alembert's principle and systems of conservative forces, ideas which form a nucleus for more advanced work, are presented in an intuitive manner, with, however, sufficient analytical background so as not to fail to impress the student. The selection of problems in both texts is to be commended. In "Applied Mechanics" a large number of problems, illustrative of those met with in technical practice, are given. "Intermediate Mechanics" contains a large number of interesting examples most of which are genuine illustrations of mechanical principles rather than mere exercises in algebra or trigo-

nometry. In the latter chapters of both texts many problems are considered which are to be found in a first course of differential equations. They are treated with enough detail so that such a course need not be required as a prerequisite.

Mechanics is not an easy subject and its inherent difficulties makes many problems desirable. In these texts the exercises have been wisely chosen or devised and the reviewer feels that the space devoted by the authors to illustrative examples will be well rewarded by the encouragement it offers to the student.

S. B. LITTAUER

Advanced Algebra. By Palmer H. Graham and F. Wallace John. Prentice Hall, Inc., New York. 1930. 257 pages. $1.85.

Perhaps the first question which arises concerning any new book in the field of college algebra is what new or added feature does the book in question possess which the many others we already have do not. This book by Graham and John is designed, the authors say, to meet the needs of the ordinary freshman class, and has been written throughout with the view of enabling the student to use the book with as little help from the instructor as possible.

The first six chapters are devoted to what the authors say may be considered review material. One of these chapters is on logarithms. It seems to the reviewer that the study of logarithms is of sufficient value to warrant a more important place in the book. A somewhat similar situation exists concerning variation, which is considered only incidentally in the study of linear equations. The solution of equations is taken up in succeeding chapters. In this sequence, however, the authors meet with difficulty in proving the remainder theorem. They state, on page 104, that $x^n - r^n$ is divisible by $x - r$, without having mentioned it or proved it previously in the book. Preceding the second proof of the theorem that an equation $f(x) = 0$ of the nth degree can not have more than n roots the authors make this statement, on p. 110: "In case all the roots are different, we can also prove the theorem as follows." It seems to the reviewer that in order to prove the theorem by the method employed, it is not necessary to specify that all roots are different for the proof would still hold even though some of the roots were equal.

In discussing the solution of a system of homogenous linear equations, it seems that the student might have some difficulty in grasping the significance of the indeterminate form 0/0. And since the desired result may be obtained in a more natural manner, it would be preferable to avoid using this form. The authors give the two symbols for permutation, ${}_nP_r$ and ${}^nP_r$, specifying they will use ${}_nP_r$. A similar specification is neglected for the symbols of combinations ${}_nC_r$ and ${}^nC_r$, for they use the former in the discussion of combinations and the latter in the chapter on the binomial theorem.

Since the text has been written with the idea in view of having the student use it with as little help as possible from the instructor, it would seem that the inclusion of an index and answers to some of the problems would be desirable.

There is a goodly number and wide variation of problems, apparently chosen with care. The explanation of all topics is given in a clear and concise manner, and simply enough for the average freshman to comprehend. The authors religiously follow their plan of presenting concrete examples before giving the formal proof of a theorem. A very commendable feature is a tabulation, at the end of the book, of definitions used in the text. The historical notes given at the end of several chapters, will be found interesting and helpful by student and instructor. The physical make-up of the book is attractive. On the whole, the reviewer believes the book to be quite teachable, and a text from which one can obtain good results.

H. M. HOSDORF

Trigonometry. By A. R. Crathorne and E. B. Lytle. Henry Holt and Company, 1930. Text: ix+199 pages; tables: xvi+95 pages. $1.96.

This text in trigonometry, although conventional with respect to its content, presents its material in such a concise and straightforward manner that its appearance is not only justified but welcomed. The authors have made a consistent and successful attempt to make their work sound pedagogically, rigid mathematically, and interesting to the student. The material of the text is nicely coördinated with the traditional course in college algebra; and at the same time it gives a good foundation in trigonometric analysis as required for the subsequent study of mathematics.

In numerous instances, after formulas are stated algebraically, they are translated or put into words to make the meaning clearer to the student. At the beginning of the chapter on logarithms is given a discussion of approximate numbers and errors. Since considerable computation is required in a course in trigonometry this explanation is sorely needed by the average freshman. The text first defines the trigonometric functions for acute angles, and makes the extensions later. This procedure helps to prevent a beginner from being overwhelmed at the start by more new material than he can assimilate.

The book offers a large and well varied set of problems, many of the practical examples being illustrated by figures. Answers are given to odd problems only, thereby satisfying all demands in this respect. In both problems and answers angles are expressed to the nearest tenth of minute, seconds being avoided. This is probably the preferable treatment if not the most common.

Although the book occasionally deviates from the straight and narrow path to make an interesting observation, the reviewer feels that the text could be improved by further references to the historical development of the subject and items of human interest. The preface contains an outline of a trigonometry course in thirty assignments. Even though this may not meet the needs of a particular institution, it at least offers a helpful suggestion.

The physical make-up of the book is quite pleasing. The binding, figures, and arrangement upon the page are all excellent. The book is convenient in size (5×8 instead of the larger and more common 6×9). However the smaller

page necessitates smaller type, which is objectionable in the tables, and somewhat so in the statements of the problems. A class of thirty students who had just completed a course of trigonometry examined the book and was very favorably impressed by it. Their only criticism concerned the size of type, mentioned above.

The book appears to be a polished product, and its many points of excellence will undoubtedly make it one of the most popular college texts in trigonometry.

WAYNE DANCER

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3475. *Proposed by Emma T. Lehmer, Brown University.*

Let $F(x_1, x_2, \cdots, x_n)$ be any integral, symmetric function of an even number, n, of variables and of degree less than n. Further let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be all the odd numbers less than $2n$ and $\mu_1, \mu_2, \cdots, \mu_n$ be all the numbers prime to 6 and less than $3n$. Finally let

$$\cos(\pi\lambda_\gamma/n) = \alpha_\gamma, \quad \cos(2\pi\mu_\gamma/(3n)) = \beta_\gamma \quad (\gamma = 1, \cdots, n).$$

Prove that

$$F(\alpha_1, \alpha_2 \cdots, \alpha_n) = F(\beta_1, \beta_2, \cdots, \beta_n).$$

3476. *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

Given the cubic, $x^3+ax+b=0$, with rational roots α, β, γ; show that the equation $\alpha y^2+\beta y+\gamma=0$ also has rational roots.

3477. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

If two points, harmonically separated by the centers of two unequal circles (spheres), are diametrically opposite on a circle (sphere) coaxial with the given circles (spheres), these two points are the centers of similitude of the two given circles (spheres).

Note: This is the converse of the property of the circle (sphere) of similitude. (See, for instance, Nathan Altshiller-Court, *College Geometry*, p. 196.)

3478. *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

A segment M_0M_n of unit length is divided into equal parts by points $M_1, M_2, \cdots, M_{n-1}$. Another segment M_0C of length $\sqrt{(p/n)}$ (p being a given constant) is drawn perpendicular to M_0M_n, and C is joined with the points of division. M_1M_2' is drawn parallel and equal to CM_2, $M_2'M_3'$ parallel and equal to CM_3, and finally $M_{n-1}'M_n'$ parallel and equal to CM_n. Find the limit of the locus of the vertices of this broken line as $n\to\infty$, and consequently $M_0C\to 0$.

UNSOLVED PROBLEMS

Solutions are desired for the following problems, which are the remaining unsolved problems proposed in 1913.

191(187) [1913, 196; 1919, 214]. *Proposed by L. E. Dickson.*

Find an amicable number triple by solving one of the equations (other than the last) in this Monthly, vol. 20 (1913), p. 92. Note that a solution a is to be excluded if not prime to the numbers in the same line.

279(274) [1913, 222; 1919, 214]. *Proposed by W. W. Landis, Dickinson College.*

A dam backs up the water for two miles. If the dam is raised 18 inches, will the water two miles up the stream be raised 18 inches, more or less?

196(192) [1913, 223; 1919, 214]. *Proposed by Charles Macauley, Chicago, Ill.*

Combinations containing an even number of letters are formed of the letters a, b, c, d, etc. It is required to place the letters in two columns, so that half the letters in every combination are placed in one column and the other letters of the combination in the other column, and so that all the a's are placed in the same column: all the b's in the same column; all the c's in the same column, etc.

348 [1913, 312; 1919, 268]. *Proposed by E. L. Dodd, University of Texas.*

Let $(x_1, x_2, \cdots, x_n)$ be a point in n dimensions lying in the "sphere" S defined by

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leqq 1.$$

Let T be that part of S defined by a set of n linear homogeneous inequalities with non-vanishing determinant; thus:

$$a_ix_1 + b_ix_2 + \cdots + k_ix_n \geqq 0, \quad i = 1, 2, \cdots, n.$$

Find the value of

$$\frac{\int_T \cdots \int dx_1 \cdots dx_n}{\int_S \cdots \int dx_1 \cdots dx_n};$$

in other words, find the magnitude of a "solid angle" in n dimensions, with the "sphere" as unit solid angle.

Note:—This problem was discussed and left unsolved by Schläfli in the Quarterly Journal of Mathematics for 1858, 1860, 1867. *Editor.*

202 [1913, 313; 1919, 312]. *Proposed by A. R. Schweitzer, Chicago, Ill.*

There exists an infinitude of systems of dyads $\{\alpha\beta\}$ in 7, 9, 11, etc., elements such that each system has the following properties: (1) if $\alpha\beta$ is in the set, $\beta\alpha$ is not in the set; (2) for each dyad $\alpha\beta$ in the set there exists an element ξ such that $\xi\beta$ and $\alpha\xi$ are also in the set. For example such a system is

12,	23,	34,	45,	56,	67,	78,	89,	91,
13,	24,	35,	46,	57,	68,	79,	81,	92,
14,	25,	36,	47,	58,	69,	71,	82,	93.
51,	62,	73,	84,	95,	16,	27,	38,	49.

Investigate the existence of

I. A finite set of triads $\{\alpha\beta\gamma\}$ such that (1) if $\alpha\beta\gamma$ is in the set, then $\beta\gamma\alpha$, $\gamma\alpha\beta$ are also in the set but $\beta\alpha\gamma$ is not in the set, (2) for each triad $\alpha\beta\gamma$ in the set there exists an element ξ such that $\xi\beta\gamma$, $\alpha\xi\gamma$, $\alpha\beta\xi$ are also in the set.

II. A finite set of tetrads $\{\alpha\beta\gamma\delta\}$ such that (1) if $\alpha\beta\gamma\delta$ is in the set, then $\beta\gamma\alpha\delta$, $\gamma\alpha\beta\delta$, $\gamma\delta\alpha\beta$ are also in the set but $\beta\alpha\gamma\delta$ is not in the set, (2) for each tetrad $\alpha\beta\gamma\delta$ in the set there exists an element ξ such that $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi$ are also in the set.

The problem for alternating n-ads for $n>4$ is obvious.

SOLUTIONS

2844 [1920, 326]. *Proposed by the late J. L. Riley, Stephenville, Texas.*

Decompose into simple fractions the number 6099380351/1271808720 (Gauss, *Disq. Arith.*, *Werke*, vol. 1, pp. 386–387).

Solution by Otto Dunkel, Washington University.

The fraction may be written

$$f = 4 + \frac{1012145471}{1271808720}.$$

The denominator is the product of the relatively prime factors $59\cdot 47\cdot 49\cdot 16\cdot 13\cdot 9\cdot 5 = 59\cdot 21556080$. By Euclid's method it will be found that

$$1012145471 = 52 \cdot 21556080 - 1843571 \cdot 59,$$

and hence

$$f = 4 + \frac{52}{59} - \frac{1843571}{21556080}.$$

We now reduce the last fraction on the right in the same manner. The successive reductions may be made by use of the following equalities:

$$\begin{aligned} 1843571 &= 107533 \cdot 47 - 7 \cdot 458640, \\ 107533 &= 27 \cdot 9360 - 2963 \cdot 49 \\ 2963 &= 368 \cdot 16 - 5 \cdot 585, \\ 368 &= 11 \cdot 13 + 5 \cdot 45, \\ 11 &= 4 \cdot 5 - 9. \end{aligned}$$

The result after a slight reduction is

$$f = 1 + \frac{52}{59} + \frac{7}{47} + \frac{22}{49} + \frac{11}{16} + \frac{5}{13} + \frac{4}{9} + \frac{4}{5}.$$

3420 [1930, 196]. *Proposed by the late G. B. M. Zerr.*

Let $x=f(t)$ be the equation giving the horizontal distance of a projectile in a resisting medium in terms of the time t. Prove that the vertical distance is given by

$$y = -gf(t)\int \frac{dt}{f'(t)} + g\int \frac{f(t)}{f'(t)}dt + Af(t) + B,$$

where A and B are constants.

Solution by William Hoover, Columbus, Ohio.

Let ds/dt be the total velocity of the projectile at any instant t, and let the resistance be $k(ds/dt)^n$, where k and n are positive constants. The equations of motion are then

$$\ddot{x} + k(ds/dt)^n dx/ds = \ddot{x} + k(ds/dt)^{n-1}\dot{x} = 0, \tag{1}$$

and

$$\ddot{y} + k(ds/dt)^{n-1}\dot{y} + g = 0. \tag{2}$$

Then (1) is satisfied by $x=f(t)=f$; and hence $\ddot{f}+k(ds/dt)^{n-1}\dot{f}=0$. Solving this last equation for $k(ds/dt)^{n-1}$ and setting its value in (2), we have

$$\ddot{y} - (\ddot{f}/\dot{f})\dot{y} + g = 0. \tag{3}$$

In order to solve this last equation, set $\dot{y}=u\dot{f}$; and then (3) becomes $\dot{u}+(g/\dot{f})$

$=0$. The integration of this equation gives $u=-g\int(dt/\dot{f})+A$, and we have then $\dot{y}=-g\dot{f}\int(dt/\dot{f})+A\dot{f}$. Finally, we obtain

$$y = -g\int \dot{f}\left[\int (dt/\dot{f})\right]dt + Af + B.$$

Integrating by parts the first term on the right, we have the required result.

3425 [1930, 260]. *Proposed by Pauline Sperry, University of California.*

Cauchy's linear equation $\sum_{i=0}^{n} c_i x^{n-i} D^{n-i} y = X(x)$, where the c_i are constants, is reduced to an equation with constant coefficients by the substitution $x=e^z$. Then $x^k D^k y = \sum_{i=0}^{k-1} a_i^{(k)} D^{k-i} y$, where $Dy=dy/dz$. Show that the $a_i^{(k)}$ build a sort of Pascal's triangle with k elements in the kth row, in which any element of the $k+1$st row may be obtained from those in the kth row by taking the two numbers diagonally above it to the left and right and subtracting k times the former from the latter with the understanding that if either of the diagonals takes one outside the triangle the corresponding term is zero.

Solution by Eugene Stephens, Washington University.

The proof of the proposition in the problem may be obtained from the known symbolic derivative formula

$$x^k D^k = \prod_{j=0}^{k-1}(\theta - j), \quad x = e^z, \tag{1}$$

where $D^k y = d^k y/dx^k$ and $\theta^i y = d^i y/dz^i$. A proof of this formula will be given. By direct differentiation we easily find

$$xD = \theta; \quad x^2 D^2 = \theta^2 - \theta = \theta(\theta - 1). \tag{2}$$

These results suggest (1); and the proof of (1) may be obtained by mathematical induction. Taking the derivative with respect to x of both sides of (1), and multiplying the result by x, we have

$$kx^k D^k + x^{k+1} D^{k+1} = \theta \prod_{j=0}^{k-1}(\theta - j).$$

Then, by the use of (1) and by the transposition of a term, we have

$$x^{k+1} D^{k+1} = \prod_{j=0}^{k}(\theta - j),$$

and this completes the proof of (1).

The expansion of the right side of (1) may be written

$$x^k D^k = \sum_{i=1}^{i=k} a_i^{(k)} \theta^i, \quad a_k^{(k)} = 1, \quad a_1^{(k)} = (-1)^{k-1}(k-1)!. \tag{3}$$

Hence

$$x^{k+1}D^{k+1} = \sum_{i=1}^{k+1} a_i^{(k+1)}\theta^i = (\theta - k)\sum_{i=1}^{k} a_i^{(k)}\theta^i;$$

and then

$$(4) \qquad a_i^{(k+1)} = -ka_i^{(k)} + a_{i-1}^{(k)}, \quad a_{k+1}^{(k+1)} = a_k^{(k)} = 1, \quad a_1^{(k+1)} = -ka_1^{(k)}.$$

The first two rows of the triangle are given by (2); and (4) enables us to write the following rows in turn, thus:

k							
1				1			
2			1		−1		
3		1		−3		2	
4	1		−6		11		−6

If we write $|a_i^{(k)}| = C_k^{(k-i)}$, the positive integers $C_k^{(k-i)}$ are the Stirling numbers of the first kind,[1] or the factorial coefficients of rank k.

Also solved by J. P. Dalton and Mabel S. Graham.

3426 [1930, 260]. *Proposed by B. C. Wong, Berkeley, California.*

Prove or disprove:

$$\sum_{i=0}^{t}(-1)^i\binom{r}{i}\frac{(2r-2i-1)!}{(r-2i)!} = (r-1)!,$$

where $t = r/2$ if r is even and $t = (r-1)/2$ if r is odd.

Solution by J. P. Dalton, University of Witwatersrand, South Africa.

This elementary binomial series may be evaluated by means of the identity

$$(1-x^2)^r(1+x)^{-r} = \sum_{i=0}^{r}(-1)^i\binom{r}{i}x^{2i}\sum_{j=0}^{\infty}(-1)^j\binom{r+j-1}{j}x^j = (1-x)^r.$$

Equate the coefficients of x^r and we have the required result.

Note by the Editors: The solver also gave a solution of 3399 [1929, 543] which is the same as II [1930, 323]. The method of solution given in III [1930, 323] applies also to this problem.

3427 [1930, 260]. *Proposed by L. S. Johnston, University of Detroit.*

Let

$$S_n \equiv \tfrac{1}{2}n(n+1), \quad C_{n-k} \equiv (-1)^{k+1}S_{n+1-|k|};$$

---

[1] Nielsen. *Handbuch der Theorie der Gammafunktion*, Teubner, 1906, p. 67. Numerous references to the literature of these numbers are there given. See also the article, *A note on Stirling's numbers*, by Ginsburg in this Monthly, vol. 35 (1928), pp. 77–80.

show that the equation

$$\sum_{k=-n}^{k=n} C_{n-k}x^{n+k} = 0$$

has no negative roots, and that the number of positive roots is two or zero according as n is odd or even.

Solution by J. P. Dalton, University of Witwatersrand, Johannesburg, South Africa.

This equation is easily seen to be the standard type reciprocal equation

$$\sum_{j=0}^{n-1}(-1)^{n+j-1}S_{j+1}(x^j + x^{2n-j}) + (-1)^{2n-1}S_{n+1}x^n = 0.$$

It has no negative roots (degree even: alternating signs). On summing the series on the left we get

$$\frac{(-1)^{n-1}}{(1+x)^3}[1 + x^{2n+3} + (-1)^n(2n+3)(x^{n+1} + x^{n+2})] = 0.$$

The numerator has $2n+3$ roots in all. Of these, three are the triple negative root introduced in the process of summing; hence there are $2n$ positive or complex roots. By the Descartes rule the number of positive roots cannot exceed two, therefore the given equation has either two positive roots, or none. If n is even

$$-[1 + x^{2n+3} + (2n+3)(x^{n+1} + x^{n+2})]$$

can have no positive root. If n is odd

$$[1 + x^{2n+3} - (2n+3)(x^{n+1} + x^{n+2})]$$

has a positive root lying between 0 and 1. Since it has one positive root, it must have two, and no more. This completes the proof.

3429 [1930, 261]. *Proposed by Paul Wernicke, Washington, D. C.*

Given a tetrahedron $ABCD$ and a point P not on one of its edges. Draw through P three lines, each meeting two opposite edges. Express the ratios in which the latter will be divided by these points of intersection in terms of four quantities a, b, c, d, the edge AB being divided in the ratio $a:b$, etc.

Solution by Vladimir F. Ivanoff, San Francisco, California.

The planes s_1, s_2, s_3 passing through P and DA, DB, and DC, respectively, have the line PD in common. Denote the intersection of PD with the plane ABC by E. The lines EA, EB and EC are then the intersections of the plane ABC with s_1, s_2, and s_3. Denote the intersection of EA and BC by M_{bc}; the intersection of EB and AC by M_{ac}; the intersection of EC and AB by M_{ab}.

Then the lines PM_{ab}, PM_{ac} and PM_{bc} pass through opposite edges of the tetrahedron.

For the segments AM_{ab}, BM_{ab}, BM_{bc}, etc. we have

$$\frac{AM_{ab}\cdot BM_{bc}\cdot CM_{ac}}{AM_{ac}\cdot BM_{ab}\cdot CM_{bc}} = 1, \tag{1}$$

where the ratio is considered positive when the point of division lies within the finite segment.

It we take

$$AM_{ab}:BM_{ab} = a:b$$

and

$$BM_{bc}:CM_{bc} = b:c,$$

it follows from (1) that

$$AM_{ac}:CM_{ac} = a:c.$$

If we take

$$AM_{ad}:DM_{ad} = a:d,$$

we can, by similar process, find the remaining ratios:

$$BM_{bd}:DM_{bd} = b:d$$

and

$$CM_{cd}:DM_{cd} = c:d.$$

3432 [1930, 314]. *Proposed by Vladimir F. Ivanoff, San Francisco, California,*

Given n points, $A_1, A_2, A_3, \cdots, A_n$ in a plane. Arbitrary lines, A_iA_j, are drawn (joining pairs of these points) provided that they do not intersect each other. Prove that, regardless of the relative positions of the given points the possible number of lines is $3(n-2)$ if $n \geqq 3$.

Solution by Lester R. Ford, The Rice Institute.

Take n points, $A_1, \cdots, A_n$, on a sphere and let paths joining them be drawn on the surface of the sphere. This is an equivalent problem (by stereographic projection, for instance) from the standpoint of Analysis Situs.

Let the maximum number of paths be drawn. An additional assumption is necessary. We may require either (1) that not more than one path shall be drawn between two points; or (2) that no path shall be continuously deformable into another path between the same end points without meeting some other path.

The given points, the paths connecting them, and the regions into which these paths separate the surface of the sphere may be looked upon as the vertices, edges, and faces, respectively, of a polyhedron; and Euler's formula,

$$V + F = E + 2,$$

is satisfied. Here $V=n$; and E is the required number of paths.

The faces of this polyhedron are triangular. Suppose, on the contrary, that there is a face, f, with four or more sides, and let A_i, A_j, A_k, A_l be consecutive vertices of f. Under assumption (2), a path A_iA_k or a path A_jA_l drawn in f is

an allowable path. If assumption (1) is made, either A_i and A_k or A_j and A_l may be already joined by a path lying outside f, but both pairs cannot be so joined since the two paths would cross; and there is again an allowable path in f. In either case, the hypothesis that the maximum number of paths has been drawn is violated.

Now F triangles have $3F$ sides. Since each edge forms a side of two triangles, we have $E=3F/2$, or $F=\frac{2}{3}E$. Substituting in Euler's formula:

$$n + \tfrac{2}{3}E = E + 2;$$

whence

$$E = 3(n - 2),$$

which was to be proved.

Also solved by A. G. Clark and Paul Wernicke.

3434 [1930, 315]. *Proposed by Harry Gwinner, Baltimore, Maryland.* Evaluate

$$\lim_{x\to 0} (1 + x)^{\log x}.$$

Solution by Ralph P. Agnew, Cornell University.

On letting y represent the expression of which the limit is to be found, we have $\log y = \log x \log (1+x)$. Elementary methods suffice to show that as $x\to 0$, $x \log x \to 0$ and $(1/x) \log (1+x) \to 1$. Hence,

$$\lim_{x\to 0} \log y = \lim_{x\to 0} [x \log x] \cdot [(1/x) \log (1 + x)] = 0.$$

Thus, the required limit is 1.

Also solved by S. F. Bibb, A. G. Clark, D. C. Duncan, Emma M. Gibson, V. F. Ivanoff, and William Orange.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor R. C. Archibald, of the department of mathematics at Brown University, has been appointed lecturer in mathematics at Harvard University for the second semester of the present academic year.

Dr. Wilhelm Blaschke, Director of the Mathematical Seminar, University of Hamburg, will lecture at the Johns Hopkins University as James Speyer Visiting Professor of Mathematics from March 2, to May 8, 1931. He will lecture on "Topological Questions in Differential Geometry" four hours weekly and will also be available for consultation by students.

Professor Harald Bohr, of the University of Copenhagen, has been appointed professor of physics at Princeton for the second half of this academic year.

Dr. W. L. Bragg, of the University of Manchester, has been appointed lecturer in mathematics at Cornell University for the year 1930–31.

Dr. Egon S. Pearson of the Biometric Laboratory, University of London, will lecture on Mathematical Statistics during the coming summer session from June 8 to July 16, inclusive, at the University of Iowa.

Professor R. C. Archibald, of Brown University, delivered his lecture on "Mathematics Prior to the Greeks," at Rutgers University on Friday evening, December 5, 1930.

Miss Helen Calkins has been appointed professor of mathematics at the Pennsylvania College for Women, Pittsburgh.

Mr. C. A. Lovell, of the Drexel Institute, has been appointed a member of the technical staff of the Bell Telephone Laboratories.

Professor Richard Morris addressed the Mathematics group of Haverford College on November 19, 1930, speaking on "Some Modern Geometry of the Triangle with reference to the Orthopole."

Assistant Professor W. C. Risselman, of the University of Pittsburgh, has been appointed professor of mathematics at the Arizona State Teachers College, Flagstaff.

Dr. C. W. Andrews, Librarian Emeritus of the John Crerar Library, died on November 20, 1930. He had been a member of the Mathematical Association since September, 1925.

Mary E. Caster, teacher of mathematics in the Eastside High School, Patterson, N. J., died on December 7, 1930. She was a charter member of the Mathematical Association.

Major C. H. Chepmell, of Bristol, England, a member of the Mathematical Association since May, 1921, died on November 18, 1930.

J. O. Eckersley, chief engineer, Department of Markets, New York City, a charter member of the Association, died on November 11, 1930.

Associate Professor R. A. Sheets, of Denison University, died in Colorado on December 3, 1930, after an illness of nearly a year. He had been a member of the Mathematical Association since December, 1921.

Professor H. E. Trefethen, a teacher of mathematics and astronomy in Colby College for nearly twenty years, died on November 3, 1930. He was a charter member of the Mathematical Association.

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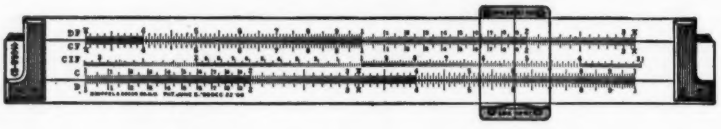

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CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fifteenth Summer Meeting of the Association, Minneapolis, Minnesota, Sept. 7-8, 1931.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1931.

ILLINOIS, Peoria, May 1-2.
INDIANA, Muncie, May 1-2.
IOWA, Davenport, May 1-2.
KANSAS, Topeka, Kansas, Jan. 24.
KENTUCKY, April 15.
LOUISIANA-MISSISSIPPI, Natchitoches, La., March 13-14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA.
MISSOURI, St. Louis, November.
NEBRASKA, Lincoln, May.
OHIO—Columbus, Ohio, April 2.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Boulder, Colo., April 17-18.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 14.
TEXAS, Fort Worth, Tex., Jan. 31.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to department or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus Ohio.

THE FIFTEENTH ANNUAL MEETING OF THE ASSOCIATION

The fifteenth annual meeting of the Mathematical Association of America was held at Cleveland, Ohio, on Wednesday and Thursday, December 31, 1930, and January 1, 1931, in affiliation with the American Association for the Advancement of Science and the American Mathematical Society. Two hundred sixty-one were in attendance at the meetings, including the following one hundred ninety-seven members of the Association:

C. R. ADAMS, Brown University
R. P. AGNEW, National Research Fellow, Poland, Ohio
O. P. AKERS, Allegheny College
J. W. ALEXANDER, Princeton University
R. B. ALLEN, Kenyon College
NOLA L. ANDERSON, Sophie Newcomb College
R. C. ARCHIBALD, Brown University
C. S. ATCHISON, Washington and Jefferson College
W. L. AYRES, University of Michigan

R. W. BABCOCK, Kansas State Agricultural College
CLARA L. BACON, Goucher College
GRACE M. BAREIS, Ohio State University
WALTER BARTKY, University of Chicago
I. A. BARNETT, University of Cincinnati
M. A. BASOCO, University of Nebraska
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